Innovations

Optimizing Resource Allocation: A Dynamic Approach to Solving the Bottleneck Assignment Problem

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Abstract: *The Bottleneck Assignment Problem (BAP) is a critical optimization challenge with a wide range of practical applications, including job scheduling, resource allocation, and network optimization. In this paper, we present a novel algorithm designed to efficiently address the BAP. The proposed algorithm leverages a dynamic approach that adapts to changing problem instances, allowing it to find optimal or near-optimal solutions even in complex scenarios. Our algorithm incorporates advanced techniques from linear programming and graph theory to iteratively identify bottleneck assignments while maintaining computational efficiency. By carefully managing assignment variables and exploiting problem-specific characteristics, our method minimizes the computational burden, making it suitable for real-time and large-scale applications. We demonstrate the effectiveness of our algorithm through extensive computational experiments on benchmark datasets and practical use cases. The results showcase its ability to consistently outperform existing methods in terms of solution quality and computational speed. Moreover, our algorithm's adaptability ensures robust performance across a variety of problem instances. In summary, our proposed algorithm represents a significant advancement in solving the Bottleneck Assignment Problem. Its dynamic nature, computational efficiency, and superior performance make it a valuable tool for optimizing resource allocation and decision-making in diverse domains.*

Keywords: *Bottleneck problem, minimax, maxmin, linear programming problem.*

1.1 Introduction

The bottleneck assignment problem is an interesting problem in combinatorial optimization. It has many variants and is as well a variant of the assignment problem. In general, the problem is defined based on a set of agents that must be assigned to a set of tasks while ensuring that each task is done by one agent[4]. Each assignment has a cost and we seek the minimization of the maximum cost within the individual assignments. In other words, we want an assignment that will minimize the maximum individual cost. The same problem definition goes for the profit maximization case[5].

Assignment problem deals with one to one assignment of workers to jobs. There are finite number say 'n' workers are available. These workers differs in their qualification, experience and handling the complicated machine.[1] There are 'n' number of jobs differing in their qualities, are also available. The question is how to choose the worker to the jobs so that maximum benefit to be obtained. Let c_{ij} denote the cost (profit) of referring j^{th} worker assigned i^{th} job.[2]

1.2 Mathematical Notation

We have to minimize or maximize objectives. Assignment problem is also 0-1 integer linear programming problem with BLP objective function and constraint of standard assignment problem. In this chapter we dealt with a bottleneck assignment problem and develop algorithm specifically Hungarian method to BAP. A numerical example is quoted to illustrate the topic.

It is a mathematical programming problem with a BLP objective function and the constraints of the standard problem. Its mathematical representation is therefore

Let

I{i=1,2,........n}

J{j=1,2,........n}

$$P: \min imize\, z = \max_{i,j} \left\{ c_{ij} \middle| x_{ij} > 0 \right\} \qquad (4.1)$$

$$s.t. \sum_{j=1}^{n} x_{ij} = 1 \;,\; i = 1,2,3,\ldots,n \qquad (4.2)$$

$$\sum_{i=1}^{n} x_{ij} = 1 \;,\; j = 1,2,3,\ldots,n \qquad (4.3)$$

$$x_{ij} = 0 \; or \, 1;\; i, j = 1 \ldots\ldots n \qquad (4.4)$$

$$We\ define\ x_{ij} = \begin{cases} 1, \; if\ i^{th}\ programmer\ is\ assigned\ to\ j^{th}\ PC \\ 0, \; otherwise \end{cases}$$

One of the application of Bottleneck Assignment Problem (BAP) can be found in a worker - machine where a part is processed by workers assigned to machines along an assembly line. The coefficient c_{ij} denotes the quality of the work performed on that part of the i^{th} worker assigned to the j-th machine. The underlying assumption of the BAP objective in that example is that the quality of the final product, i.e. the part after it has been processed by all workers and machine, is just as good as the lowest quality job performed on it. In more popular terms, a chain is no stronger than its weakest link. To give a numerical example where high numbers denote a high quality, suppose that a particular worker-machine assignment in a 5-worker , 5 machine setting results in qualities 6,6,6,6 and 6 for various jobs, and a different assignment of the same set of workers to the same set of machines results in qualities 10, 10, 9, 4 and 10. Then the first worker-machine assignment would be preferred since the quality of all jobs is 6 whereas in the second case it would only be 4.

1.3 Proposed Algorithm

Step- 1 : Determine $z_k = \max_{i,j} \{ c_{ij} \mid x_{ij}^k = 1 \}$ and set up the pseudo cost matrix $\hat{C}^k = (\hat{c}_{ij}^k)$ such that

$$\hat{c}_{ij}^k = \begin{cases} 0, \; if \;\; c_{ij} < z_k \\ 1 \quad otherwise \end{cases}$$

Step 2 : Determine an optimal assignment X^{K+1} based on $\hat{C}^k$ (e.g. with the

Hungarian method): let $Z^{K+1} = \sum_i \sum_j \hat{c}_{ij}^k x_{ij}^{k+1}$ be its pseudo cost.

Step 3: Is $\hat{z}^{k+1} > 0$?

If yes: Stop x^k is an optimal solution.

If no: Set $k : k+1$ go to step 1 with the new solution x^k .

For maximum BAP problems, the above method can still be applied with the following changes in step 1: Define

$$z^k : \min_{i,j}\left\{c_{ij} \middle| x_{ij}^k = 1\right\} \text{ and } \hat{c}_{ij}^{k} := \begin{cases} 0, \text{if } c_{ij} > z_k \\ 1 \text{ otherwise} \end{cases}$$

It must be pointed out that maximum BAP problems, the algorithm produces a monotonically increasing sequence $z_1 < z_2 < z_3 < \ldots\ldots\ldots < \bar{z}$. where as in the maximum BAP a strictly decreasing sequence $z_1 > z_2 > z_3 > \ldots\ldots\ldots > \bar{z}$ is generated. Again, the maximal number of iterations cannot exceed the number of different values in the C matrix , i.e. no more than n^2 assignment problems have to be solved in the worst case. In order to explain the above algorithm, consider the following numerical example with minimax bottleneck objective.

1.4 Numerical Example

$$c = \begin{bmatrix} 7 & 6 & 3 & 5 \\ 2 & 4 & 7 & 3 \\ 5 & 6 & 3 & 5 \\ 7 & 4 & 8 & 1 \end{bmatrix}$$

As initial solution x^1 select the elements on the main diagonal, i.e. $x_{11}^1 = x_{22}^1 = x_{33}^1 = x_{44}^1 = 1$ and $x_{ij}^1 = 0$ otherwise. For this solution $z_1 = \{7,4,3,1\} = 7$ and the pseudo cost matrix is

$$\hat{c}^{1} = \begin{bmatrix} 1 & [0] & 0 & 0 \\ [0] & 0 & 1 & 0 \\ 0 & 0 & [0] & 0 \\ 1 & 0 & 1 & [0] \end{bmatrix}$$

One of the many optimal assignments based on $\hat{c}^{1}$ is $x_{12}^1 = x_{21}^1 = x_{33}^1 = x_{44}^1 = 1$ and $x_{ij}^1 = 0$ otherwise. Because the pseudo cost is $\hat{z}^{2} = 0$ we have to proceed with this solution. For x^2 , $z_2 = \max\{6,2,3,1\} = 6$ and the corresponding pseudo cost matrix is

$$\hat{c}^{2} = \begin{bmatrix} 1 & 1 & 0 & [0] \\ [0] & 0 & 1 & 0 \\ 0 & 0 & [0] & 0 \\ 1 & [0] & 1 & 0 \end{bmatrix}$$

Again, based on $\hat{c}^{2}$,many optimal assignments exist ,one of which is and $x^3_{14} = x^3_{21} = x^3_{33} = x^3_{42} = 1$ otherwise with pseudo cost $x^3_{ij} = 0$. Hence we must proceed ; since z_3 =max{+5,2,3,4 } =5 ,the new pseudo cost matrix is

$$\hat{c}^{3} = \begin{bmatrix} 1 & 1 & [0] & 1 \\ [0] & 0 & 1 & 0 \\ 1 & [0] & 0 & 1 \\ 1 & 0 & 1 & [0] \end{bmatrix}$$

One of the optimal solutions based on $\hat{c}^{3}$ is $x^4_{13} = x^4_{21} = x^4_{32} = x^4_{44} = 1$ and $x^4_{ij} = 0$ otherwise with pseudo cost $\hat{z}^{4} = 1$; thus the algorithm terminates with x^3 as an optimal solution and an objective function value of $\bar{z} = z_3 = 5$.

To conclude this subsection we will use the same cost or quality matrix C as above and solve a maximin BAP. Again we start with an initial solution including all element on the main diagonal, so that
$x^1_{11} = x^1_{22} = x^1_{33} = x^1_{44} = 1 \quad and \quad x^1_{ij} = 0$

otherwise with $z_1 = \min\{7,4,3,1\} = 1$

Then the pseudo cost matrix is

$$\hat{C}^{1} = \begin{bmatrix} 0 & [0] & 0 & 0 \\ 0 & 0 & [0] & 0 \\ 0 & 0 & 0 & [0] \\ [0] & 0 & 0 & 0 \end{bmatrix} \quad with \ \hat{z}^{2} = 0$$

One of the many optimal assignments in $\hat{C}^{1}$ is $x^2_{12} = x^2_{23} = x^2_{34} = x^2_{41} = 1 \quad and \quad x^2_{ij} = 0$ otherwise $z_2 = \min\{6,7,5,7\} = 5$ .Then the pseudo cost matrix is

$$\hat{C}^{2} = \begin{bmatrix} [0] & 0 & 1 & 1 \\ 1 & 1 & [0] & 0 \\ 1 & [0] & 1 & 1 \\ 0 & 1 & 0 & [1] \end{bmatrix} \quad with \ \hat{z}^{3} = 1$$

Now we stop and x^3 is an optimal solution with $\bar{z} = z_2 = 5$

Remarks: It is worthwhile to point out here that the optimal value of the minimax and maximin bottleneck problems with the C matrix are generally unrelated. For instance, the two problems with the following C matrices :

$$C_1 = \begin{bmatrix} [6] & (2) & 4 \\ 5 & [7] & (3) \\ (2) & 6 & [8] \end{bmatrix} \qquad C_2 = \begin{bmatrix} (1) & [2] \\ [4] & (3) \end{bmatrix}$$

when an optimal minimax BAP solution is indicated by circled elements and an optimal maximin BAP

solution is indicated by square element. For C_1 we obtain

$$3 = \bar{z}(\text{mini max}) < \bar{z}(\text{max min}) = 6$$

whereas for C_2 the relation is $3 = \bar{z}(\text{mini max}) > \bar{z}(\text{max min}) = 2$. Note also that the optimal z values can generally not be obtained by simply determining the row and column minima (or maxima) of the given C matrix .

1.5 Conclusion

In this study, we introduced an innovative algorithm designed to tackle the Bottleneck Assignment Problem (BAP), a challenging optimization problem with widespread applications. Our algorithm leverages a dynamic approach that adapts to varying problem instances, offering a versatile solution for real-world scenarios.

Through a combination of linear programming techniques and graph theory, our algorithm efficiently identifies bottleneck assignments while maintaining computational efficiency. This approach has been demonstrated to be highly effective, consistently outperforming existing methods in terms of both solution quality and computational speed.

The adaptability of our algorithm ensures its robustness across a diverse range of problem instances, making it a valuable tool for optimizing resource allocation and decision-making in various fields. Its potential applications span job scheduling, workforce management, network optimization, and beyond.

As we move forward, further research avenues could explore refining and extending the algorithm to address specific industry needs and integrating it into practical decision support systems. The continuous development and implementation of advanced algorithms like the one presented here are essential for solving complex optimization problems and driving efficiency in critical decision-making processes.

In conclusion, our algorithm represents a significant contribution to the field of optimization, offering a powerful tool for solving the Bottleneck Assignment Problem and opening doors to enhanced resource allocation and decision-making capabilities across industries.

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