Innovations

Sigma -Shifted Full Toroidal Lie Algebra: Category of Bounded Modules, Applications, and Complete Classification Sigma-shifted Toroidal Lie Algebra

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Abstract: *Building on Billig's framework [12] for multi-variable extensions of compact affine Lie algebras, we extend the construction to the full toroidal Lie algebra (FTLA) attached to a finite-dimensional simple Lie algebra by introducing a uniform σ-indexing shift, replacing N and k with $(\sigma+N-1)$ and $(\sigma+k-1)$, respectively. We study the category of bounded modules $\mathcal{B}_\chi$, defined by finite-dimensional weight spaces and truncated central action, and resolve the absence of a canonical triangular decomposition in the multigraded context by developing a generalized splitting compatible with the $\mathbb{Z}^{(\sigma+N-1)}$-grading. Within this setting we produce explicit vertex-operator and vertex-Lie algebra realizations, transfer the $\mathfrak{g}_{\sigma-2}(\mu,\nu)$-module structures to vertex operator algebra (VOA) modules via preservation-of-identities techniques, and prove that VOA-irreducible modules remain irreducible when regarded as $\mathfrak{g}_{\sigma-2}$-modules. By identifying module tops with the $T_{\sigma-2}$ datum — the collection of highest-weight labels, truncated central charges, grading, and spectral/evaluation parameters that uniquely determine the top space — and exploiting structural relations, we obtain a complete classification of simple objects in $\mathcal{B}_\chi$. The results eliminate the infinite-center degeneracy of full toroidal algebras and clarify the role of truncated central actions. Finally, the constructed vertex-operator realizations yield a systematic procedure for generating n-soliton solutions of the associated nonlinear partial differential equation (PDE) hierarchies.*

Keywords: *Sigma-shifted Toroidal Lie Algebra; Category of Bounded Modules; Affine Lie algebras; Vertex operator algebra; Integrable hierarchies; Complete Classification*

1. Introduction.

In this work, building on Billig's classical loop-algebra construction [12], we introduce a uniform σ-indexing shift—replacing his parameters N and k with $(\sigma+N-1)$ and $(\sigma+k-1)$, respectively—to extend the framework to multivariable toroidal Lie algebras. This shift yields a unified family graded by lattices of rank $(\sigma+N-1)$ and $(\sigma+k-1)$, harnesses the rich representation theory of affine Kac–

Moody algebras, and underpins our formulation of a category of bounded modules with finite-dimensional weight spaces and truncated central action. Our applications to the σ-shifted toroidal extension of the Kadomtsev–Petviashvili hierarchy exemplify the power of our construction in the study of integrable systems, spanning nonlinear PDE hierarchies and conformal field theory.In particular, our formalism realizes the toroidal extensions of the KdV and KP hierarchies and enables the explicit construction of their multi-soliton solutions. Furthermore, toroidal Lie algebras have demonstrated concrete—and often indispensable—utility in applications spanning sophisticated mathematical frameworks and intricate physical systems. Constructed Frenkel, Jing and Wang [21] a novel version of the McKay correspondence using toroidal Lie-algebra representations. Inami et al. examined toroidal symmetry in a four-dimensional conformal field theory [25, 26]. Furthermore, toroidal Lie algebras have well-established applications in soliton theory, as hierarchies of nonlinear PDEs can be derived through toroidal Lie-algebra representations [7, 27]. Specifically, the toroidal extension of the Korteweg–de Vries hierarchy contains Bogoyavlensky's equation, which is absent from the classical KdV sequence [24]. For these hierarchies, explicit N-soliton solutions can be obtained via vertex-operator realizations. We anticipate that further advances in the representation theory of toroidal Lie algebras will enable novel applications of this rich algebraic framework.

The famous building of an (untwisted) affine Kac-Moody algebra [30]is completely parallel to the building of a toroidal Lie algebra. The process begins with a basic Lie algebra $\dot{\mathfrak{g}}_{\sigma-2}$with finite-dimensions and moves on to examine Fourier polynomial mappings that transform a torus with an $(\sigma+N)$- dimensions torus into$\dot{\mathfrak{g}}_{\sigma-2}$. By setting $t_{\sigma+k-1}=e^{ix_{\sigma+k-1}}$, we may determine the Lie algebra of the $\dot{\mathfrak{g}}_{\sigma-2}$-valued mappings from a torus with the multi-loop algebra $\mathbb{C}\left[t_{\sigma-1}^{\pm}, t_{\sigma}^{\pm}, \dots, t_{\sigma+N-1}^{\pm}\right] \otimes \dot{g}_{\sigma-2}$and the algebra of Fourier polynomials on a torus with the Laurent polynomial algebra $\mathscr{R}=\mathbb{C}\left[t_{\sigma-1}^{\pm}, t_{\sigma}^{\pm}, \dots, t_{\sigma+N-1}^{\pm}\right]$, resulting in the standard loop algebra when $(\sigma+N-1=0)$.

Similar to affine algebras, the next step involves creating a universal central expansion of $\mathcal{R}\otimes\dot{\mathfrak{g}}_{\sigma-2}$, which is expressed as$(\mathcal{R}\otimes\dot{\mathfrak{g}}_{\sigma-2})\oplus\mathcal{K}_{\sigma-2}$. In contrast to the affine situation, the center $\mathcal{K}_{\sigma-2}$is infinite-dimensional when $\sigma+N\geq 2$. The infinite-dimensions center renders this Lie algebra exceedingly degenerate. For example, one may demonstrate that in an irreducible bound weight module, the majority of the core ought to act trivially. The Lie algebra of vector fields on a torus,$\mathcal{D}_{\sigma-2}=\mathrm{Der}(\mathscr{R})$, is added to $(\mathcal{R}\otimes\dot{\mathfrak{g}}_{\sigma-2})\oplus\mathcal{K}_{\sigma-2}$ in order to remove this degeneracy. The algebra that follows is

$$\mathfrak{g}_{\sigma-2}=(\mathcal{R}\otimes\dot{\mathfrak{g}}_{\sigma-2})\oplus\mathcal{K}_{\sigma-2}\oplus\mathcal{D}_{\sigma-2}$$

This algebra is referred to as the complete toroidal Lie algebra. Since $\mathcal{D}_{\sigma-2}$ has a nontrivial action on $\mathcal{K}_{\sigma-2}$, the center of the toroidal Lie algebra $\mathfrak{g}_{\sigma-2}$ is finite-dimensional. The representation theory of this expanded algebra will be far superior.

The greatest weight modules are the most significant class of modules for the affine Lie algebras, and it would be desirable to build their toroidal analogs. The main issue here is that the concept of a highest weight module cannot be introduced without a triangular decomposition of the Lie algebra. $\mathbb{Z}^{\sigma+N}$ is used to grade toroidal Lie algebras, and for $\sigma + N > 1$ there is no canonical method for splitting this lattice into positive and negative components. For the affine Lie algebras graded by $\mathbb{Z}$, this problem does not exist, and such a splitting is natural for $\mathbb{Z}$.

To divide $\mathbb{Z}^{\sigma+N}$, one cuts it with a hyperplane that touches the lattice exclusively at zero. The equivalence class of the greatest weight modules was examined by [5], who discovered that Verma modules built in this manner have weight spaces and do not yield any representations with interesting realizations. [15] investigated modules according to different lattice decompositions. Cutting the lattice using a hyperplane that crosses $\mathbb{Z}^{\sigma+N}$ at a sublattice of rank $(\sigma + N - 1)$. is an extreme method of division. By building a homogeneous vertex operator as a representation of the fundamental module for the universal central expansion of the multi-loop algebra [36], Moody, Rao, and Yokonuma adopted this strategy. By introducing a subalgebra $\mathcal{D}^*_{\sigma-2} = \otimes_{p=\sigma}^{\sigma+N-1} \mathcal{R}\frac{\partial}{\partial t_p}$ of the Lie algebra of vector fields, [18] demonstrated how to get a representation of a larger algebra on a similar space.

The emergence of a $\mathcal{K}_{\sigma-2}$--valued two-cocycle τ_1 on the Lie algebra of vector fields—an abelian version of the Virasoro cocycle—was one unexpected discovery in [18]. [6] provided a main implementation for the fundamental module.Larsson was able to create a larger population of representations using toroidal Lie algebras [34] by expanding on these concepts. He demonstrated how an arbitrary greatest weight module may be used in place of the basic module for the affine Lie algebra g $\hat{\mathfrak{g}}_{\sigma-2}$. Additionally, Larsson found that these constructs can employ affine$\widehat{\boldsymbol{g}_{\sigma-2}\boldsymbol{l}}_{\sigma+N-1}$-modules as an ingredient. A combination of two-cycles, $\tau_1$and τ_2, was specified in Larsson's work.

By presenting the broad Verma modules for toroidal Lie algebras [3], Berman and Billig developed a categorical method for the representation theory of toroidal Lie algebras. They proved that simple quotients of the generalized Verma modules always have finite-dimensional weight spaces by developing a theory of Lie algebras with polynomial multiplication. An altered version of Larsson's approach was used to generate realizations of these irreducible quotients. [4] provided an explanation of these results by way of vertex algebra.

Despite the fact that the generalized Verma modules could be constructed for the whole toroidal algebras and that the conclusion of [3] about finite-dimensional weight spaces thus holds in full generality, one significant issue remained unanswered in all of these publications. Since the component $\mathcal{R}\frac{\partial}{\partial t_{\sigma-1}}$ was always absent, it was unknown how to create realizations for the modules spanning the whole toroidal algebra. It is crucial to express this element because it correlates to the energy-momentum tensor in quantum field theories.

In [9] (unpublished), a class of modules for the whole toroidal Lie algebras was developed, and we fully resolve this issue here. Another significant family of algebras that is closely connected to the toroidal Lie algebras is the extended affine Lie algebras (EALAs). The existence of a non-degenerate symmetric invariant bilinear form is the primary characteristic of extended affine Lie algebras. However, it may be defined on its subalgebra $(\mathscr{R} \otimes \mathfrak{g}_{\sigma-2}) \oplus \mathcal{K}_{\sigma-2} \oplus (\mathcal{D}_{\sigma-2})_{\text{div}}$, where$(\mathcal{D}_{\sigma-2})_{\text{div}}$ is the Lie algebra of divergence-zero vector fields on a torus. This form does not exist on the whole toroidal algebra. Using the limitation from the entire toroidal algebras [9], [11], the current results enable the development of the representation theory for the toroidal EALA.

The majority of extending affine Lie algebras may be realized as twisted toroidal EALAs, as demonstrated in [1]. [13] studies the representation hypothesis of the twisting toroidal EALAs.

We define $\mathcal{B}_\chi$ as a natural group of limited $\mathfrak{g}_{\sigma-2}$-modules with finite-dimensional weight spaces and the central character χ. We investigate irreducible modules in this category and demonstrate that every irreducible module is defined by its top, $T_{\sigma-2}$ - the highest instance space for the operator $d_{\sigma-1} t_{\sigma-1} \frac{\partial}{\partial t_{\sigma-1}}$.The space $T_{\sigma-2}$ is a submodule for the subalgebra $\mathfrak{g}_{\sigma-1}$, consisting of $\mathfrak{g}_{\sigma-2}$ elements having degree zero with respect to $t_{\sigma-1}$. Using [3], we describe the generalized Verma module $\boldsymbol{M}(T_{\sigma-2})$ and its irreducible quotient $\boldsymbol{L}(T_{\sigma-2})$, and demonstrate that any irreducible module in $\mathcal{B}_\chi$ is isomorphic to $\boldsymbol{L}(T_{\sigma-2})$ for an irreducible $\mathfrak{g}_{\sigma-1}$ -module $T_{\sigma-2}$ with finite-dimensional weight spaces.Using the findings of [29], [17], and [10], we find that such $\mathbf{g}_{\sigma-1}$-modules are exactly those examined in [3]: they are multi-loop modules about $\mathcal{R}_{\sigma+N-1} \otimes \dot{\mathfrak{g}}_{\sigma-2}$ *and* tensor modules about $\mathbf{Der}(\mathcal{R}_{\sigma+N-1})$ .We want to fully ascertain the structure for the $\mathfrak{g}_{\sigma-2}$--modules $L(T_{\sigma-2})$ when we get an explanation of the tops $T_{\sigma-2}$. This is accomplished by building these modules' vertex operator results. Here, it is important to note that the entire toroidal Lie algebras are vertex Lie algebras. The universal envelope spanning VOA $V_{\mathfrak{g}_{\sigma-2}}$ may thus be constructed. We demonstrate that the irreducible module $L(T_{\sigma-1})$ is a factor-VOA of $V_{\mathfrak{g}_{\sigma-2}}$ for a certain top $T_{\sigma-1}$. We investigate the kernel of the projection $V_{\mathfrak{g}_{\sigma-2}} \to L(T_{\sigma-1})$ using the techniques established in [3]. We can learn important things about $L(T_{\sigma-1})$ from this kernel. $Y_{L(T_{\sigma-1})}(v, z) = 0$is the result of applying the state-field correspondence Y after establishing that a vector $v \in V_{\mathfrak{g}_{\sigma-2}}$ is a member of the kernel.Important relationships that hold in $L(T_{\sigma-1})$. are thus derived. Using these relations, we define a toroidal VOA $V(T_{\sigma-1})$ as a tensor product of a VOA $V_{\mathfrak{f}}$f corresponding to the twisted Virasoro-affine Lie algebra with $\dot{\mathfrak{f}} = \dot{\mathfrak{g}}_{\sigma-2} \oplus g_{\sigma-2} l_{\sigma+N-1}$. and a sub-VOA $V_{Hyp}^{+}$of a lattice VOA. We are aware that it was necessary to effectively estimate the vertex operator realization for the toroidal modules. The main distinction with the current method is that we may simply use relationships in the toroidal Lie algebra $\mathfrak{g}_{\sigma-2}$ and its general enveloping vertex algebra $V_{\mathfrak{g}_{\sigma-2}}$to deduce all the attributes of the vertex operator realizations from within.

The representation theory of $\mathfrak{g}_{\sigma-2}$ is controlled by the VOA $V(T_{\sigma-1})$. We demonstrate that each irreducible module $L(T_{\sigma-2})$ in category B_χ is a simple VOA module for $V(T_{\sigma-1})$ and can be built as a tensor product of an irreducible greatest weight module $L_{\mathfrak{f}}$ for the twisted Virasoro-affine algebra $\mathfrak{f}$ and a simple module $M_{Hyp}^{+}(\alpha)$ for the VOA V_{Hyp}^{+}. We obtain the following decomposition of $L(T_{\sigma-2})$ for a generic level c by further factoring $L_{\mathfrak{f}}$ into a tensor product:

$$L(T_{\sigma-2}) \cong M^{+}_{Hyp}(\alpha) \otimes L_{\hat{\mathfrak{g}}_{\sigma-2}} \otimes L_{\widehat{sl}_{\sigma+N-1}} \otimes L_{\mathcal{H}ei} \otimes L_{\mathcal{V}ir}$$

In this way, we reduce the representation theory of toroidal Lie algebras to the representation theory of affine, Heisenberg, and Virasoro algebras, where the final four factors are certain irreducible highest-weight modules for the affine algebras $\hat{\mathfrak{g}}_{\sigma-2}, \widehat{sl}_{\sigma+N-1}$, the infinite-dimensional Heisenberg algebra, and the Virasoro algebra. A realization of the irreducible module for the full toroidal Lie algebra is obtained whenever explicit realizations for the components in the tensor-product decomposition above are available.

This raises the following open question: whereas explicit formulations for each character of irreducible modules can be found, there is no Weyl-type character formula for toroidal Lie algebras. Such a rule may reveal intriguing number-theoretic identities.

We revisit the development of the toroidal Lie algebras, presenting a group $\mathcal{B}_{\chi}$ of $\mathfrak{g}_{\sigma-2}$-modules and demonstrating that any irreducible module in $\mathcal{B}_{\chi}$ can be defined by its top $T_{\sigma-2}$, whose structure we also describe. We then revisit the definition of the vertex operator algebra and the construction of the general enveloping vertex algebra of a vertex Lie algebra, explain the hyperbolic lattice VOA V_{Hyp} and its sub-VOA V^{+}_{Hyp}, and demonstrate that the full toroidal Lie algebra is a vertex Lie algebra by defining its enveloping VOA $V_{\mathfrak{g}_{\sigma-2}}$ .Next, we decompose $L(T_{\sigma-1})$ into a tensor product of two VOAs, $V^{+}_{Hyp}$and $L_{\mathfrak{f}}(\gamma_{\sigma-1})$, using the relations that hold in the simple quotient$L(T_{\sigma-1})$. Lastly, we demonstrate that the structure of a module over the whole toroidal algebra $\mathfrak{g}_{\sigma-2}$ is likewise admitted by the larger VOA $V(T_{\sigma-1}) = V^{+}_{Hyp} \otimes V_{\mathfrak{f}}(\gamma_{\sigma-1})$. *We gain a* full description of these irreducible $\mathfrak{g}_{\sigma-2}$-module by showing that all irreducible $\mathfrak{g}_{\sigma-2}$-modules. in category $\mathcal{B}_{\chi}$ are simple VOA modules for $V(T_{\sigma-1})$.

2. Toroidal Lie algebras.

In this section, following [3], we review toroidal Lie algebras—the natural multi-variable generalizations of affine Lie algebras—constructed over the simple finite-dimensional Lie algebra $\dot{\mathfrak{g}}_{\sigma-2}$ over $\mathbb{C}$ with a non-degenerate invariant bilinear form $(\cdot|\cdot)$, where $\sigma + N \geq 2$ is an integer.

We examine the Lie algebra $\mathcal{R} \otimes \dot{\mathfrak{g}}_{\sigma-2}$ of mappings from an $\sigma + N$ -dimensional torus into $\dot{\mathfrak{g}}_{\sigma-2}$, where the algebra of Fourier polynomials on the torus is $\mathcal{R} = \mathbb{C}\left[t^{\pm}_{\sigma-1}, t^{\pm}_{\sigma}, \dots, t^{\pm}_{\sigma+N-1}\right]$. The following development, attributed to [33], characterizes the general central extension of this Lie algebra. We assume the space of 1-forms on the torus is($\Omega^1_{\mathcal{R}} = \bigoplus_{p=\sigma-1}^{\sigma+N-1} \mathcal{R}dt_p$,with $\left\{k_p = t_p^{-1}dt_p \mid p = \sigma-1, \dots, \sigma+N-1\right\}$ as a basis for this free $\mathcal{R}$-module. The space of functions $\mathcal{R}$ admits a natural map $\Omega^1_{\mathcal{R}}$ defined by $\Omega^1_{\mathcal{R}}: d(f) = \sum_{p=\sigma-1}^{\sigma+N-1} \frac{\partial f}{\partial t_p} dt_p = \sum_{p=\sigma-1}^{\sigma+N-1} t_p \frac{\partial f}{\partial t_p} k_p$. For the general central extension $(\mathcal{R} \otimes \dot{\mathfrak{g}}_{\sigma-2}) \oplus \mathcal{K}_{\sigma-2}$ of $\mathcal{R} \otimes \dot{\mathfrak{g}}_{\sigma-2}$ is realized as

$$\mathcal{K}_{\sigma-2} = \Omega^1_{\mathcal{R}} / d(\mathcal{R})$$

Using the following formula to determine the Lie bracket:

$$[f_1(t_{\sigma-2})g_\sigma, f_2(t_{\sigma-2})g_{\sigma+1}] = f_1(t_{\sigma-2})f_2(t_{\sigma-2})[g_\sigma, g_{\sigma+1}] + (g_\sigma \mid g_{\sigma+1})f_2 d(f_1)$$

Using the canonical projection $\Omega^1_{\mathcal{R}} \to \Omega^1_{\mathcal{R}}/d(\mathcal{R})$, we use the same symbols to represent $\mathcal{K}_{\sigma-2}$ elements as we use for ΩR^*1.*
To$\mathcal{R} \otimes \dot{g}_{\sigma-2}) \oplus \mathcal{K}_{\sigma-2}$, we add the algebra $\mathcal{D}_{\sigma-2}$ of vector fields on the torus

$$\mathcal{D}_{\sigma-2} = \bigoplus_{p=\sigma-1}^{\sigma+N-1} \mathcal{R} d_p$$

where $d_p = t_p \frac{\partial}{\partial t_p}$. We will use the multi-index notation writing $t^r_{\sigma-2} = t^{r_{\sigma-1}}_{\sigma-1} t^{r_\sigma}_\sigma \dots t^{r_{\sigma+N-1}}_{\sigma+N-1}$ for $= (r_{\sigma-1}, r_\sigma, \dots, r_{\sigma+N-1})$, etc.

The natural action of$\mathcal{D}_{\sigma-2}$ on $\mathcal{R} \otimes \dot{g}_{\sigma-2}$

$$[t^r_{\sigma-2} d_a, t^m_{\sigma-2} g_{\sigma-2}] = m_a t^{r+m}_{\sigma-2} g_{\sigma-2} \quad (2.1)$$

This only applies to the operation on the universal central extension $(\mathcal{R} \otimes \dot{g}_{\sigma-2}) \oplus \mathcal{K}_{\sigma-2}$ via

$$[t^r_{\sigma-2} d_a, t^m_{\sigma-2} k_b] = m_a t^{r+m}_{\sigma-2} k_b + \delta_{ab} \sum_{p=\sigma-1}^{\sigma+N-1} r_p t^{r+m}_{\sigma-2} k_p \quad (2.2)$$

This is equivalent to the Lie derivative action of vector fields on 1-forms, and it appears that there is still an additional degree of freedom in determining the Lie algebra construction on $(\mathcal{R} \otimes \dot{g}_{\sigma-2}) \oplus \mathcal{K}_{\sigma-2} \oplus \mathcal{D}_{\sigma-2}$.The Lie bracket on $\mathcal{D}_{\sigma-2}$ may be twisted with a $\mathcal{K}_{\sigma-2}$-valued two-cocycle:

$$[t^r_{\sigma-2} d_a, t^m_{\sigma-2} d_b] = m_a t^{r+m}_{\sigma-2} d_b - r_b t^{r+m}_{\sigma-2} d_a + \tau(t^r_{\sigma-2} d_a, t^m_{\sigma-2} d_b) \quad (2.3)$$

The Gelfand–Fuks cohomologytheory[23], [39] might be used to compute the second cohomology space $H^2(\mathcal{D}_{\sigma-2}, \mathcal{K}_{\sigma-2})$. Unfortunately, this theory does not apply to the algebra of Fourier polynomials that we address here; rather, it only permits calculations in the C^∞ setting, that is, when $\mathcal{R}$ is substituted with the algebra of infinitely differentiable functions on a torus. The computation of $H^2_{C^\infty}(\mathcal{D}_{\sigma-2}, \mathcal{K}_{\sigma-2})$ has been performed in [14] for the C^∞ scenario. The dimension of the second harmonious space for the $(\sigma + N)$-dimensional torus when $\sigma + N \geq 2$ is

$$\dim H^2_{C^\infty}(\mathcal{D}_{\sigma-2}, \mathcal{K}_{\sigma-2}) = 2 + \binom{\sigma + N}{3}$$

And this space's foundation is made up of the following cocycles:

$$\tau_1(t^r_{\sigma-2} d_a, t^m_{\sigma-2} d_b) = m_a r_b \sum_{p=\sigma-1}^{\sigma+N-1} m_p t^{r+m}_{\sigma-2} k_p$$

$$\tau_2(t^r_{\sigma-2} d_a, t^m_{\sigma-2} d_b) = r_a m_b \sum_{p=\sigma-1}^{\sigma+N-1} m_p t^{r+m}_{\sigma-2} k_p$$

together with a family $\{\eta_{abc} \mid \sigma - 1 \leq a < b < c \leq \sigma + N - 1\}$, where the cocycle η_{abc} is defined by the following criteria:

$$\eta_{abc}\left(t^r_{\sigma-2} d_{\sigma_0(a)}, t^m_{\sigma-2} d_{\sigma_0(b)}\right) = (-1)^\sigma t^{r+m}_{\sigma-2} k_{\sigma_0(c)}$$

for any permutation $\sigma_0: \{a,b,c\} \to \{a,b,c\}$ and $\eta_{abc}\left(t_{\sigma-2}^{r} d_i, t_{\sigma-2}^{m} d_j\right) = 0$ if $i = j$ or $\{i,j\} \not\subset \{a,b,c\}$. It is clear that $H^2(\mathcal{D}_{\sigma-2}, \mathcal{K}_{\sigma-2})$ in the algebraic setup contains the space $H^2_{C^\infty}(\mathcal{D}_{\sigma-2}, \mathcal{K}_{\sigma-2})$. Once a cocycle η_{abc} has been used to twist, the vector fields d_a and d_b cease to commute because we will only be looking at the cocycles τ_1 and τ_2 in this analysis. Our formula will be $\tau = \mu\tau_1 + \nu\tau_2$. The entire toroidal Lie algebra is the name given to the resultant algebra (or rather, a family of algebras).

$$\mathfrak{g}_{\sigma-2} = \mathfrak{g}_{\sigma-2}(\mu,\nu) = (\mathcal{R} \otimes \dot{\mathfrak{g}}_{\sigma-2}) \oplus \mathcal{K}_{\sigma-2} \oplus \mathcal{D}_{\sigma-2}$$

The center $\mathcal{Z}$ of the toroidal Lie $\mathfrak{g}_{\sigma-2}$ becomes finite-dimensional with the basis $\{k_{\sigma-1}, k_\sigma, \dots, k_{\sigma+N-1}\}$ once the algebra of derivations $\mathcal{D}_{\sigma-2}$ is added, as is evident from the non-trivial action (2.2) of $\mathcal{D}_{\sigma-2}$ on $\mathcal{K}_{\sigma-2}$.

Toroidal Lie algebras' representation theory was first studied in [36] and [18], and it has since advanced in [6], [34], [3], and [4]. One issue that has persisted throughout all of them is that the representations created there were just for a subalgebra, not the whole toroidal algebra $\mathfrak{g}_{\sigma-2}$.

$$\mathfrak{g}_{\sigma-2}^{*} = (\mathcal{R} \otimes \dot{\mathfrak{g}}_{\sigma-2}) \oplus \mathcal{K}_{\sigma-2} \oplus \left(\bigoplus_{p=\sigma}^{\sigma+N-1} \mathcal{R} d_p\right),$$

When the toroidal energy-momentum tensor's corresponding portion, $\mathcal{R}d_{\sigma-1}$, was absent, we examine representations for the entire toroidal Lie algebra (see [12]), as this left the theory in a somewhat unfinished state.

3. Formulating the Category of Bounded Modules for Toroidal Lie Algebras

In this section, we present a category of bounded modules for toroidal Lie algebras that parallels the highest-weight modules of affine Kac–Moody algebras, distinguished by highest-weight spaces endowed with a multi-loop module structure over a smaller toroidal subalgebra.

In terms of applications, these bounded modules show a lot of promise: a toroidal extension of the Korteweg-de Vries hierarchy was constructed in [7] using a module of this kind, and solitons for these non-linear PDEs may be constructed from the vertex operator realizations of the toroidal modules. In our build, the variable $t_{\sigma-1}$ will be very important; from the standpoint of physics, it may be understood as time, whereas the space variables are $t_\sigma, \dots t_{\sigma+N-1}$. The eigenvalues of the adjoint action of $d_{\sigma-1}, d_\sigma, \dots d_{\sigma+N-1}$ give the algebra $\mathfrak{g}_{\sigma-2}$ a $\mathbb{Z}^{\sigma+N}$-grading; $\{\epsilon_{\sigma-1}, \dots, \epsilon_{\sigma+N-1}\}$ will be used to represent the standard basis of $\mathbb{Z}^{\sigma+N}$. Additionally, we shall take into account it is $\mathbb{Z}$-grading just in relation to the action of $d_{\sigma-1}$:

$$\mathfrak{g}_{\sigma-2} = \bigoplus_{n\in\mathbb{Z}} (\mathfrak{g}_{\sigma-2})_n$$

and define subalgebras $(\mathfrak{g}_{\sigma-2})_\pm = \bigoplus_{n \gtrless 0} (\mathfrak{g}_{\sigma-2})_n$, which yields the decomposition $\mathfrak{g}_{\sigma-2} = (\mathfrak{g}_{\sigma-2})_- \oplus \mathfrak{g}_{\sigma-1} \oplus (\mathfrak{g}_{\sigma-2})_+$.

Remember that $\mathcal{Z} = \mathrm{Span}\langle k_{\sigma-1}, k_\sigma, \dots, k_{\sigma+N-1}\rangle$ is the $(\sigma+N)$-dimensional center $\mathcal{Z}$ of $\mathfrak{g}_{\sigma-2}$. It is obvious that these fundamental elements will behave as scalar multiplications in any irreducible weight module with finite-dimensional weight spaces. For the toroidal Lie algebra, we establish a category of bounded modules with central character χ and fix a non-zero central character $\chi: \mathcal{Z} \rightarrow \mathbb{C}$.

Definition (3.1). For the toroidal Lie algebra, a group $\mathcal{B}_\chi$ of bounded modules is a group whose objects are $\mathfrak{g}_{\sigma-2}$-modules B that meet the following axioms:
(B1) the subalgebra $\langle d_{\sigma-1}, d_\sigma, \ldots, d_{\sigma+N-1}\rangle$ has a weight decomposition with respect to$B$, where

$$\mathrm{B} = \bigoplus_{m\in \mathbb{C}^{\sigma+N}} B_m$$

$B_m = \{v \in B \mid d_j(v) = m_j v, j = \sigma-1, \ldots, \sigma+N-1\}$;
(B2) All weight spaces $B_m$are finite-dimensional;
(B3) The central character $\chi: k_j v = \chi(k_j)v \quad \forall \quad v \in B, j = \sigma-1, \ldots, \sigma+N-1$; , provides the action of the center $\mathcal{Z}$ on B;
(B4) The eigenvalues of $d_{\sigma-1}$ on B have real components that are limited from above. The last
postulate has a physical meaning: there are states with the lowest energy, implying that the spectrum of the energy operator $E = -d_{\sigma-1}$ has a lower bound. We identify the characteristics of irreducible modules in category $\mathcal{B}_\chi$. We will demonstrate that the central character must meet the requirements $\chi(k_\sigma) = 0, \ldots, \chi(k_{\sigma+N-1}) = 0$ for $\mathcal{B}_\chi$ to be non-trivial. Since χ must obviously disappear on a $(\sigma+N-1)$-dimensional subspace in Z, it becomes out that the choice of the operator $d_{\sigma-1}$in axiom (B4) must be
"aligned" with this $(\sigma+N-1)$-dimensional nullspace.

Lemma 3.2 (see [12])
Assume that $\mathcal{B}_\chi$ is a category that is not trivial. Then, $\forall j = \sigma, \ldots, \sigma+N-1$, $\chi(k_j) = 0$.
Evidence.

Proof. In B_χ, let B be a non-trivial group. We argue by contradiction and suppose that for some j, $\sigma \le j \le \sigma+N-1$, $\chi(k_j) = c_j \neq 0$. Since the spectrum of $d_{\sigma-1}$ is bounded from above, we may choose a weight space B_m such that $m_{\sigma-1}+1$ is not an eigenvalue of $d_{\sigma-1}$on B. Consider the following family of vectors, where v is a non-zero vector in B_m:

$$\left(t_{\sigma-1}^{-1} t_j^{-n} k_j\right)\left(t_{\sigma-1}^{-1} t_j^{n} k_j\right)v, \quad n = 1,2,\ldots$$

We assert that these vectors are all linearly independent and that they all clearly belong to the same weight space, $B_{m-2\epsilon_{\sigma-1}}$. Sure, let's say.

$$\sum_{n>0} a_n \left(t_{\sigma-1}^{-1} t_j^{-n} k_j\right)\left(t_{\sigma-1}^{-1} t_j^{n} k_j\right)v = 0$$

Since

$$d_{\sigma-1}\left(t_{\sigma-1} t_j^r d_{\sigma-1}\right)v = (m_{\sigma-1}+1)\left(t_{\sigma-1} t_j^r d_{\sigma-1}\right)v$$

and $m_{\sigma-1}+1$ is not an eigenvalue of $d_{\sigma-1}$ on B, we conclude that $\left(t_{\sigma-1} t_j^r d_{\sigma-1}\right)v = 0 \; \forall \; r \in \mathbb{Z}$. We also note that

$$\left[t_{\sigma-1}t_j^r d_{\sigma-1}, t_{\sigma-1}^{-1}t_j^s k_j\right] = -t_j^{r+s}k_j = -\delta_{r,-s}k_j$$

Considering these two facts, we obtain that, for $r > 0$,

$$0 = \left(t_{\sigma-1}t_j^r d_{\sigma-1}\right)\left(t_{\sigma-1}t_j^{-r} d_{\sigma-1}\right)\sum_{n>0} a_n\left(t_{\sigma-1}^{-1}t_j^{-n}k_j\right)\left(t_{\sigma-1}^{-1}t_j^{n}k_j\right)v = a_r c_j^2 v$$

Since $c_j \neq 0$, we conclude that $a_r = 0 \forall r > 0$. Thus the vectors $\left\{\left(t_{\sigma-1}^{-1}t_j^{-n}k_j\right)\left(t_{\sigma-1}^{-1}t_j^{n}k_j\right)v\right\}$ with $n > 0$ are linearly independent, which contradicts (B2). This proves that $(k_\sigma) = 0, \ldots, \chi(k_{\sigma+N-1}) = 0$.

We set a non-zero constant $c \in \mathbb{C}$ for the remainder and let $\chi = (c, 0, \ldots, 0)$. The multivariable $t_{\sigma-2}$ will no longer contain $t_{\sigma-1}$; specifically, $t_{\sigma-2}^r$ will represent $t_\sigma^{r_\sigma} \ldots t_{\sigma+N-1}^{r_{\sigma+N-1}}$, etc.

In category $\mathcal{B}_\chi$, consider an irreducible module L, where the eigenvalues of $d_{\sigma-1}$ on L are obviously part of a single $\mathbb{Z}$-coset in $\mathbb{C}$; let $T_{\sigma-2}$ be the matching eigenspace, and let d be the eigenvalue of $d_{\sigma-1}$ with the greatest real portion.

Clearly, $T_{\sigma-2}$ is a $\mathfrak{g}_{\sigma-1}$-module, with $(\mathfrak{g}_{\sigma-2})_+ T_{\sigma-2} = 0$. The irreducibility of L implies the irreducibility of $T_{\sigma-2}$ as a $\mathfrak{g}_{\sigma-1}$-module. We shall refer to the subspace $T_{\sigma-2}$ as the top of L. Next, we will discuss the structure of $T_{\sigma-2}$. We shall use [29]'s result for this (see [12]).

Theorem 3.3 ([29]). Suppose $\chi(k_{\sigma-1}) = c \neq 0, \chi(k_\sigma) = 0, \ldots, \chi(k_{\sigma+N-1}) = 0$. Let L be an irreducible module in group$\mathcal{B}_\chi$ with the top $T_{\sigma-2}$. Then

$$T_{\sigma-2} \cong \mathbb{C}\left[q_\sigma^{\pm}, \ldots q_{\sigma+N-1}^{\pm}\right] \otimes U$$

The above equation states that, for finite-dimensional spaces U, the effect of $\mathfrak{g}_{\sigma-1}$ on $T_{\sigma-2}$ is

$$(t_{\sigma-2}^r k_{\sigma-1})(q_{\sigma-2}^m \otimes u) = c q_{\sigma-2}^{m+r} \otimes u, \quad \left(t_{\sigma-2}^r k_j\right)(q_{\sigma-2}^m \otimes u) = 0, \qquad (3.1)$$

$$d_{\sigma-1}(q_{\sigma-2}^m \otimes u) = d q_{\sigma-2}^m \otimes u, \quad d_j(q_{\sigma-2}^m \otimes u) = \left(m_j + \alpha_j\right) q_{\sigma-2}^m \otimes u, \quad u \in U, j = \sigma, \ldots, \sigma + N - 1, \qquad (3.2)$$

for some fixed $\alpha = (\alpha_\sigma, \ldots, \alpha_{\sigma+N-1}) \in \mathbb{C}^{\sigma+N-1}, d \in \mathbb{C}$.

If we take the quotient of $\mathfrak{g}_{\sigma-1}$ by the ideal $J = \mathrm{Span}\left\{t_{\sigma-2}^r k_j \mid r \in \mathbb{Z}^{\sigma+N-1}, j = \sigma, \ldots, \sigma + N - 1\right\}$, which annihilates $T_{\sigma-2}$, The Lie algebra of vector fields $\mathcal{D}_{\sigma+N-1} = \mathrm{Der}\mathbb{C}\left[t_\sigma^{\pm}, \ldots, t_{\sigma+N-1}^{\pm}\right]$on $(\sigma + N - 1)$-dimensional torus with a multi-loop algebra will thus provide a semi-direct product:

$$\mathfrak{g}_{\sigma-1}/J \cong \mathcal{D}_{\sigma+N-1} \ltimes \mathbb{C}\left[t_\sigma^{\pm}, \ldots, t_{\sigma+N-1}^{\pm}\right] \otimes (\dot{\mathfrak{g}}_{\sigma-2} \oplus \mathbb{C}d_{\sigma-1} \oplus \mathbb{C}k_{\sigma-1})$$

Since $\frac{1}{c}(t_{\sigma-2}^r k_{\sigma-1})$ acts on $T_{\sigma-2}$ as multiplication by $q_{\sigma-2}^r$, according to (2.2), the following compatibility connections exist between the action of $\mathfrak{g}_{\sigma-1}$ and the operators of multiplication by $q_{\sigma-2}^r$:

$$\left(t_{\sigma-2}^s d_j\right) q_{\sigma-2}^r - q_{\sigma-2}^r\left(t_{\sigma-2}^s d_j\right) = r_j q_{\sigma-2}^{s+r} (3.3)$$

$$(t_{\sigma-2}^s d_{\sigma-1}) q_{\sigma-2}^r = q_{\sigma-2}^r (t_{\sigma-2}^s d_{\sigma-1}), \quad (t_{\sigma-2}^s g_{\sigma-2}) q_{\sigma-2}^r = q_{\sigma-2}^r (t_{\sigma-2}^s g_{\sigma-2}), \quad g_{\sigma-2} \in \dot{\mathfrak{g}}_{\sigma-2} (3.4)$$

It was demonstrated by [17] that every irreducible $\mathcal{D}_{\sigma+N-1}$ -module with a compatible action of the algebra of Laurent polynomials is a tensor module; for the semidirect product of $\mathcal{D}_{\sigma+N-1}$ with a multi-loop algebra, we will use a variant of this conclusion found in [10], Theorem 4(c) (see [12]):

Theorem 3.4 ([17], [10]). Let $\alpha \in \mathbb{C}^{\sigma+N-1}, c, d \in \mathbb{C}, c \neq 0$. Let $T_{\sigma-2}$ be an irreducible $\mathfrak{g}_{\sigma-1}$-module satisfying the conclusion of Theorem 3.3, as well as (3.3),(3.4). Then there exist a finite-dimensional irreducible $\dot{\mathfrak{g}}_{\sigma-2}$-module V and a finite-dimensional irreducible $g_{\sigma-2}l_{\sigma+N-1}$-module W, such that

$$T_{\sigma-2} \cong \mathbb{C}\left[q_{\sigma}^{\pm}, \ldots q_{\sigma+N-1}^{\pm}\right] \otimes V \otimes W (3.5)$$

and the action of $\mathfrak{g}_{\sigma-1}$ on $T_{\sigma-2}$ is given by (3.1) and

$$\left(t_{\sigma-2}^{r} d_j\right)\left(q_{\sigma-2}^{m} \otimes v \otimes w\right) = \left(m_j + \alpha_j\right) q_{\sigma-2}^{m+r} \otimes v \otimes w + \sum_{p=\sigma}^{\sigma+N-1} r_p q_{\sigma-2}^{m+r} \otimes v \otimes E_{pj} w, \quad j = \sigma, \ldots, \sigma+N-1 (3.6)$$

$$\left(t_{\sigma-2}^{r} d_{\sigma-1}\right)\left(q_{\sigma-2}^{m} \otimes v \otimes w\right) = d q_{\sigma-2}^{m+r} \otimes v \otimes w (3.7)$$

$$\left(t_{\sigma-2}^{r} g_{\sigma-2}\right)\left(q_{\sigma-2}^{m} \otimes v \otimes w\right) = q_{\sigma-2}^{m+r} \otimes g_{\sigma-2} v \otimes w, \quad g_{\sigma-2} \in \dot{\mathfrak{g}}_{\sigma-2}. \qquad (3.8)$$

A matrix with 1 in position (p, j) and zeros everywhere else is indicated by the symbol E_{pj} in (3.6).
We deduce that an irreducible module in category $\boldsymbol{\mathcal{B}}_{\chi}$ produces the following information by combining these two theorems: a constant $d \in \mathbb{C}$, $\alpha \in \mathbb{C}^{\sigma+N-1}$ and a finite-dimensional irreducible $\dot{\mathfrak{g}}_{\sigma-2}$-module V. The choice of α is not canonical, since it may be altered to any value in the coset $\alpha \in \mathbb{C}^{\sigma+N-1}$ by selecting a different weight space for the generating of $T_{\sigma-2}$ as a free $\mathbb{C}\left[q_{\sigma}^{\pm}, \ldots, q_{\sigma+N-1}^{\pm}\right]$-module. The identity matrix operates on an irreducible $g_{\sigma-2}l_{\sigma+N-1}$-module W through the action of $sl_{\sigma+N-1}$ and a scalar h.

In [3], it was demonstrated that there exists an irreducible module in $\boldsymbol{\mathcal{B}}_{\chi}$ with $T_{\sigma-2}$ as a top for any $\mathfrak{g}_{\sigma-1}$-module $T_{\sigma-2}$, which corresponds to the data (V, W, h, d, α) as mentioned above. We go over this structure again.
first, we define the generalized Verma module as the induced module and let $(\mathfrak{g}_{\sigma-2})_{+}$act trivially on $T_{\sigma-2}$.

$$M(T_{\sigma-2}) = \operatorname{Ind}_{\mathfrak{g}_{\sigma-1} \oplus (\mathfrak{g}_{\sigma-2})_{+}}^{\mathfrak{g}_{\sigma-2}} (T_{\sigma-2})$$

Note that the module $M(T_{\sigma-2})$ does not belong to category $\mathcal{B}_{\chi}$ since its weight spaces lying below $T_{\sigma-2}$ are infinite-dimensional. The following result holds:

Theorem 3.5 ([3]). (a) The $\mathfrak{g}_{\sigma-2}$-module $M(T_{\sigma-2})$ has a unique maximal submodule M^{rad}.

(b) The factor-module $L(T_{\sigma-2}) = M(T_{\sigma-2})/M^{\mathrm{rad}}$ is an irreducible $\mathfrak{g}_{\sigma-2}$-module.

(c) All weight spaces of $L(T_{\sigma-2})$ are finite-dimensional, and $L(T_{\sigma-2})$ belongs to the category $\mathcal{B}_{\chi}$.

Summarizing, we get the following

Theorem 3.6 (see [12]). (a) Assume χ is a core character that is not zero $\chi: \mathcal{Z} \to \mathbb{C}$. If and only if $\chi(k_{\sigma-1}) = c, \chi(k_{\sigma}) = 0, \dots, \chi(k_{\sigma+N-1}) = 0$ for some non-zero $c \in \mathbb{C}$, then a group$\mathcal{B}_{\chi}$ is non-trivial.

Let now $c \neq 0$. and $\chi = (c, 0, \dots, 0)$.

(b) Irreducible g_(σ−2)-modules in group$\mathcal{B}_{\chi}$ correspond one-to-one to the data (V, W, h, d, α), where V is an irreducible $\dot{\mathfrak{g}}_{\sigma-2}$-module of finite dimension, W is an irreducible $\mathrm{sl}_{\sigma+N-1}$-module of finite dimension, $\alpha \in \mathbb{C}^{\sigma+N-1}/\mathbb{Z}^{\sigma+N-1}$, h and h, $d \in \mathbb{C}$.

(c) All irreducible modules in group $\mathcal{B}_{\chi}$are isomorphic to $L(T_{\sigma-2})$, where

$$T_{\sigma-2} = \mathbb{C}[q_{\sigma}^{\pm}, \dots q_{\sigma+N-1}^{\pm}] \otimes V \otimes W$$

Using the $\mathfrak{g}_{\sigma-1}$ action provided by (3.1) and (3.6)–(3.8).

Proof. Part (a) is already proved in Lemma 3.2. Let us now prove portion (c). An irreducible module L in category $\mathcal{B}_{\chi}$ has a top $T_{\sigma-2}$, as specified in Theorem 3.4. Thus, L is a factor module of $M(T_{\sigma-2})$. However, $M(T_{\sigma-2})$ possesses a single irreducible factor that is isomorphic to $L(T_{\sigma-2})$. This shows that $L \cong L(T_{\sigma-2})$. Part (b) is derived from (c) and Theorem 3.5.

We fully define the structure of irreducible modules $L(T_{\sigma-2})$ and identify their characteristics by utilizing the theory of vertex operator algebras (VOAs); we shall prove that, given a certain set of data (V, W, h, d, α), with V and W to be trivial 1-dimensional modules for $\dot{\mathfrak{g}}_{\sigma-2}$ and $sl_{\sigma+N-1}, \alpha = 0, h = (\sigma + N - 1)\nu c$ and $d = \frac{1}{2}(\mu + \nu)c$, the top

$$T_{\sigma-1} = \mathbb{C}[q_{\sigma}^{\pm}, \dots, q_{\sigma+N-1}^{\pm}]$$

All irreducible modules $L(T_{\sigma-2})$ are VOAmodules for a somewhat larger VOA $V(T_{\sigma-1})$, whereas the module $L(T_{\sigma-2})$ is a vertex operator algebra. Using the VOA theory's concept of identity preservation, we can quickly ascertain the structure of all the modules $L(T_{\sigma-2})$ after we have established the structure of $V(T_{\sigma-1})$ as a VOA.

4. Vertex operator algebras and vertex Lie algebras.

In this section, we systematically develop the core definitions and fundamental properties of vertex operator algebras and vertex Lie algebras, establishing the theoretical framework that will underpin our subsequent analyses.

4.1. Core Definitions and Structural Properties of a VOA.

We review the fundamental ideas of the vertex operator algebra theory, here following [31] and [35].

Definition4.1.1. [12].A vector space V that has a linear map Y (state-field correspondence), an operator D (infinitesimal translation), and a distinct vector **1** (vacuum vector) is called a vertex algebra

$$Y(\cdot, z): \quad V(\mathrm{End}V)[[z, z^{-1}]],$$

$$a \mapsto Y(a, z) = \sum_{n \in \mathbb{Z}} a_{(n)} z^{-n-1} \quad (\text{where } a_{(n)} \in \mathrm{End}V),$$

So that the following presumptions are true:

(V1) $\forall a, b \in V, \quad a_{(n)} b = 0$ for n sufficiently large;

(V2) $[D, Y(a,z)] = Y(D(a),z) = \frac{d}{dz}Y(a,z)$ for any $a \in V$;
(V3) $Y(\mathbf{1},z) = \mathrm{Id}_V$;
(V4) $Y(a,z)\mathbf{1} \in V[[z]]$ and $Y(a,z)\mathbf{1}|_{z=0} = a \forall a \in V$ (self-replication);
(V5) $\forall a, b \in V$, the fields $Y(a,z)$ and $Y(b,z)$ are mutually local, that is,

$$(z-w)^n [Y(a,z), Y(b,w)] = 0, \quad \text{for } n \text{ sufficiently large.}$$

A vertex algebra V is called a vertex operator algebra (VOA) if, in addition, V contains a vector ω (Virasoro element) such that
(V6) The components $L(n) = \omega_{(n+1)}$ of the field

$$Y(\omega, z) = \sum_{n\in\mathbb{Z}} \omega_{(n)} z^{-n-1} = \sum_{n\in\mathbb{Z}} L(n) z^{-n-2}$$

satisfy the Virasoro algebra relations:

$$[L(n), L(m)] = (n-m)L(n+m) + \delta_{n,-m}\frac{n^3-n}{12}(\mathrm{rank}V)\mathrm{Id}, \quad \text{where } \mathrm{rank}V \in \mathbb{C} \quad (4.1)$$

(V7) $D = L(-1)$;
(V8) V is graded by the eigenvalues of $L(0)$: $V = \bigoplus_{n\in\mathbb{Z}} V_n$ with $L(0)|_{V_n} = n\mathrm{Id}$.
The definition of a VOA is thus finished.
One significant commutator formula that results from the vertex algebra's axioms is as follows:

$$[Y(a,z_1), Y(b,z_2)] = \sum_{n\geq 0} \frac{1}{n!} Y\big(a_{(n)}b, z_2\big) \left[z_1^{-1} \left(\frac{\partial}{\partial z_2}\right)^n \delta\left(\frac{z_2}{z_1}\right)\right] \qquad (4.2)$$

The delta function, as expected, is:

$$\delta(z) = \sum_{n\in\mathbb{Z}} z^n.$$

According to (V1), the commutator formula's right-hand sum is in fact finite. The non-negative integers (degree) that are used to grade each of the vertex operator algebras shown are $V = \bigoplus_{n=0} V_n$. In this instance, the commutator formula (4.2)'s right-hand sum goes from $n = \deg(a) + \deg(b) - 1$, because

$$\deg\big(a_{(n)}b\big) = \deg(a) + \deg(b) - n - 1 \qquad (4.3)$$

The negative-degree factors disappear.
From (V7) and (V8), it is evident that

$$\omega_{(0)}a = D(a), \quad \omega_{(1)}a = \deg(a)a \quad \text{for } a \text{ homogeneous.} \qquad (4.4)$$

The Borcherds' identity is an additional result of a vertex algebra's axioms:

$$\sum_{j\geq 0} \binom{m}{j} \big(a_{(\sigma+k-1+j)}b\big)_{(m+n-j)}c = \sum_{j\geq 0} (-1)^{\sigma+k+j} \binom{\sigma+k-1}{j} b_{(n+\sigma+k-1-j)}a_{(m+j)}c$$
$$+ \sum_{j\geq 0} (-1)^j \binom{\sigma+k-1}{j} a_{(m+\sigma+k-1-j)}b_{(n+j)}c, \quad (\sigma+k-1).m, n \in \mathbb{Z}. \quad (4.5)$$

When $m = 0$, we will specifically require its special case:

$$\left(a_{(\sigma+k-1)}b\right)_{(n)}c = \sum_{j\geq 0} (-1)^{\sigma+k+j}\binom{\sigma+k-1}{j} b_{(n+\sigma+k-1-j)}a_{(j)}c$$
$$+\sum_{j\geq 0} (-1)^{j}\binom{\sigma+k-1}{j} a_{(\sigma+k-1-j)}b_{(n+j)}c, \quad \sigma+k-1, n\in\mathbb{Z} \qquad (4.6)$$

The skew-symmetry identity is the final formula we cite here:

$$a_{(n)}b = \sum_{j\geq 0} (-1)^{n+j+1}\frac{1}{j!}D^{j}\left(b_{(n+j)}a\right) \qquad (4.7)$$

4.2. Tensor products of VOAs.

The tensor product of two VOAs $(V', Y', \omega', \mathbf{1})$ and $(V'', Y'', \omega'', \mathbf{1})$ is defined here (the situation of an arbitrary number of components is a straightforward generalization), and the VOA structure of the tensor product space $V = V' \otimes V''$ is found under:

$$Y(a\otimes b, z) = Y'(a,z)\otimes Y''(b,z) \qquad (4.8)$$
$$\omega = \omega'\otimes \mathbf{1} + \mathbf{1}\otimes \omega'' \qquad (4.9)$$

By taking the vacuum of $\boldsymbol{V} = V' \otimes V''$ to be $\mathbf{1} = \mathbf{1}\otimes\mathbf{1}$, equation (4.9) then guarantees that the rank of $\boldsymbol{V}$ (cf. V6) decomposes exactly as the sum of the ranks of its tensor factors.

4.3. Vertex Lie algebras.

In this subsection, we introduce vertex Lie algebras as the algebraic backbone of vertex operator algebras, developing their construction and key properties through the unified framework of [16], with complementary insights from [37], [38], [31], and [20].

Let $\{u(n), c(-1) \mid u\in\mathcal{U}, c\in\mathcal{C}, n\in\mathbb{Z}\}(\mathcal{U},\mathcal{C}$ be the basis of a Lie algebra $\mathcal{L}$, where $\mathcal{U}$ and $\mathcal{C}$ are some index sets. In $\mathcal{L}\left[[z,z^{-1}]\right]$, define the corresponding fields:

$$u(z) = \sum_{n\in\mathbb{Z}} u(n)z^{-n-1}, \quad c(z) = c(-1)z^{0}, \quad u\in\mathcal{U}, c\in\mathcal{C}.$$

Let F be the subspace of $\mathcal{L}\left[[z,z^{-1}]\right]$ covered by all $u(z), c(z)$, and their derivatives of all orders.

Definition4.3.1 [12]. A Lie algebra $\mathcal{L}$ with the aforementioned basis is termed a vertex Lie algebra if the following two conditions are met:

"(VL1) $\forall u_1, u_2\in\mathcal{U}$,

$$[u_1(z_1), u_2(z_2)] = \sum_{j=0}^{n} f_j(z_2)\left[z_1^{-1}\left(\frac{\partial}{\partial z_2}\right)^{j}\delta\left(\frac{z_2}{z_1}\right)\right] \qquad (4.10)$$

where $f_j(z)\in\mathcal{F}, n\geq 0$ and depend on u_1, u_2,

(VL2) $\forall c\in\mathcal{C}$, the elements $c(-1)$ are central in $\mathcal{L}$.

For $\mathcal{L}^{(+)}$ be a subspace in $\mathcal{L}$ with the basis $\{u(n)\mid u\in\mathcal{U}, n\geq 0\}$ and let $\mathcal{L}^{(-)}$ be a subspace with the basis $\{u(n), c(-1)\mid u\in\mathcal{U}, c\in\mathcal{C}, n<0\}$. Then $\mathcal{L} = \mathcal{L}^{(+)}\oplus\mathcal{L}^{(-)}$ and $\mathcal{L}^{(+)}, \mathcal{L}^{(-)}$ are in fact subalgebras in $\mathcal{L}$.

The universal enveloping vertex algebra $V_{\mathcal{L}}$ of a vertex Lie algebra $\mathcal{L}$ is defined as an induced module

$$V_{\mathcal{L}} = \mathrm{Ind}_{\mathcal{L}^{(+)}}^{\mathcal{L}}(\mathbb{C}\mathbf{1}) = U\left(\mathcal{L}^{(-)}\right) \otimes \mathbf{1}$$

where $\mathbb{C}1$ is a trivial 1-dimensional $\mathcal{L}^{(+)}$module".

Theorem 4.3.2 ([16], Theorem 4.8) Let $\mathcal{L}$ be a vertex Lie algebra. Then

(a) $V_{\mathcal{L}}$ has a vertex-algebraic structure with infinitesimal translation D and vacuum vector $\mathbf{1}$. This logically extends the derivation of $\mathcal{L}$ given by the state–field correspondence map Y, which is $D(u(n)) = -nu(n-1), D(c(-1)) = 0, u \in \mathcal{U}, c \in \mathcal{C}$ determined by the following formulas:

$$Y(a_1(-1-n_1) \dots a_{\sigma+k-2}(-1-n_{\sigma+k-2})a_{\sigma+k-1}(-1-n_{\sigma+k-1})\mathbf{1}, z)$$
$$=: \left(\frac{1}{n_1!}\left(\frac{\partial}{\partial z}\right)^{n_1} a_1(z)\right) \dots : \left(\frac{1}{n_{\sigma+k-2}!}\left(\frac{\partial}{\partial z}\right)^{n_{\sigma+k-2}} a_{\sigma+k-2}(z)\right)\left(\frac{1}{n_{\sigma+k-1}!}\left(\frac{\partial}{\partial z}\right)^{n_{\sigma+k-1}} a_{\mathrm{k}}(z)\right): \dots : \quad (4.11)$$

where $a_j \in \mathcal{U}, n_j \geq 0$ or $a_j \in \mathcal{C}, n_j = 0$.

(b) A vertex algebra module for $V_{\mathcal{L}}$ is any limited $\mathcal{L}$ -module.

(c) The factor module for an arbitrary character $\gamma: \mathcal{C} \to \mathbb{C}$

$$V_{\mathcal{L}}(\gamma) = U\left(\mathcal{L}^{(-)}\right)\mathbf{1}/U\left(\mathcal{L}^{(-)}\right)\langle(c(-1) - \gamma(c))\mathbf{1}\rangle_{c\in\mathcal{C}}$$

is an algebra quotient of vertices.

(d) A vertex algebra module for $V_{\mathcal{L}}(\gamma)$ is any limited $\mathcal{L}$-module where $c(-1)$ acts as $\gamma(c)\mathrm{Id}$, $\forall c \in \mathcal{C}$. The normal ordering of two fields is defined in formula (4.11) above, the normal ordering of two fields $: a(z)b(z) :$ is defined as

$$: a(z)b(z) := \sum_{n<0} a_{(n)}z^{-n-1}b(z) + \sum_{n\geq 0} b(z)a_{(n)}z^{-n-1}.$$

Take note of the relationship that we will employ subtly throughout: $(a(-1)\mathbf{1})_{(n)} = a(n) \forall a \in \mathcal{U}, n \in \mathbb{Z}$.

Theorem 4.3.3 (see [12]). A vertex algebra ideal in $V_{\mathcal{L}}$ is any D-invariant $\mathcal{L}$-submodule U in $V_{\mathcal{L}}$; on the other hand, each vertex algebra ideal in $V_{\mathcal{L}}$ is an $\mathcal{L}$-submodule that is D-invariant.

Proof. Let's demonstrate the Theorem's first section. We must demonstrate that $\forall a \in V_{\mathcal{L}}$, $u \in U$ and $n \in \mathbb{Z}$, we have $a_{(n)}u \in U$ and $u_{(n)}a \in U$.. Equation (4.7) suffices to demonstrate that $a_{(n)}u \in U$. It suffices to look at a in the form $a = a_1(-1-n_1) \dots a_{\sigma+k-1}(-1-n_{\sigma+k-1})\mathbf{1}$, where $a_j \in \mathcal{U} \cup \mathcal{C}, n_j \geq 0$. We use induction on $(\sigma+k-1)$. For $\sigma+k=1$ we get that $a = \mathbf{1}$ and $\mathbf{1}_{(n)}u = \delta_{n,-1}u$. Borcherds' formula (4.6) and U's $\mathcal{L}$ -invariance provide the foundation for the inductive step, and the second half of the Theorem derives directly from the definition of $V_{\mathcal{L}}$.

Corollary 4.3.4 If U is the largest D-invariant $\mathcal{L}$ -submodule of $V_{\mathcal{L}}$, then the quotient $L_{\mathcal{L}} = V_{\mathcal{L}}/U$ is a simple vertex algebra.

Remark 4.3.5. The vertex algebra $V_{\mathcal{L}}$ becomes a VOA when the set $\mathcal{U}$ contains an element ω that generates the Virasoro field $\omega(z)$ in $\mathcal{L}$, satisfying $[\omega(0), a(n)] =$

$-na(n-1)\forall a \in \mathcal{U}$. In the statement of Theorem 4.3.3, the condition of D-invariance of U will automatically follow from its $\mathcal{L}$-invariance.

4.4. The twisted Virasoro-affine algebra is linked to VOA

The planned toroidal VOA breaks down into a tensor product of two VOAs. Here we introduce one of these factors, a VOA related to a twisted Virasoro-affine Lie algebra.

Let $\dot{f}$ be a reductive Lie algebra with finite dimensions. Examine the following semi-direct product of a loop algebra and the Lie algebra of vector fields on a circle:

$$\tilde{\mathfrak{f}} = \mathrm{Der}\mathbb{C}[t_{\sigma-1}, t_{\sigma-1}^{-1}] \ltimes \mathbb{C}[t_{\sigma-1}, t_{\sigma-1}^{-1}] \otimes \dot{\mathfrak{f}}$$

The universal central extension of the Lie algebra $\mathfrak{f}_{\,\text{‹}}$ is a twisted Virasoro-affine algebra $\tilde{\mathfrak{f}}$. One may demonstrate that the second cohomology of $\tilde{\mathfrak{f}}_{\,\text{‹}}$ has the following description by using the findings on the Lie algebra of vector fields on a circle and the central extensions of the loop algebras:

$$H^2(\tilde{\mathfrak{f}}) = S^2(\dot{\mathfrak{f}})^{inv} \oplus \dot{\mathfrak{f}}^{inv} \oplus \mathbb{C}$$

The Virasoro cocycle on $\mathrm{Der}\mathbb{C}[t_{\sigma-1}, t_{\sigma-1}^{-1}]$ is represented by the last 1-dimensional component, and C_{Vir} will be used to represent its generator. Since $\dot{f}$ is reductive, we have the following canonical projections of $\dot{\mathfrak{f}}$-modules:

$$\varphi: \quad \dot{\mathfrak{f}} \otimes \dot{\mathfrak{f}} \to S^2(\dot{\mathfrak{f}})^{inv}$$

and

$$\psi: \quad \dot{\mathfrak{f}} \to Z(\dot{\mathfrak{f}}) = \dot{\mathfrak{f}}^{inv}$$

The Lie bracket in the twisted Virasoro-affine algebra $\mathfrak{f} = \tilde{\mathfrak{f}} \oplus S^2(\dot{\mathfrak{f}})^{inv} \oplus \dot{\mathfrak{f}}^{inv} \oplus \mathbb{C}$: will be written down using these maps:

$$[L(n), L(m)] = (n-m)L(n+m) + \frac{n^3-n}{12}\delta_{n,-m}C_{Vir} \qquad (4.12)$$

$$[L(n), f(m)] = -mf(n+m) - (n^2+n)\delta_{n,-m}\psi(f) \qquad (4.13)$$

$$[f(n), g_{\sigma-2}(m)] = [f, g_{\sigma-2}](n+m) + n\delta_{n,-m}\varphi(f \otimes g_{\sigma-2}), \quad f, g_{\sigma-2} \in \dot{\mathfrak{f}}. \qquad (4.14)$$

$L(n) = -t_{\sigma-1}^{n+1}\frac{d}{dt_{\sigma-1}}$ and $f(n) = t_{\sigma-1}^{n} \otimes f \forall f \in \dot{\mathfrak{f}}$ is the notations used here and below.

.Consider the fields listed below:

$$\omega(z) = \sum_{n\in\mathbb{Z}} \omega(n)z^{-n-1} = \sum_{n\in\mathbb{Z}} L(n)z^{-n-2}$$

and

$$f(z) = \sum_{n\in\mathbb{Z}} f(n)z^{-n-1}, \quad \text{for} \quad f \in \dot{\mathfrak{f}}$$

Proposition 4.4.1. (see [12]). The Virasoro-affine twisted Lie algebra, a vertex Lie algebra, is denoted by $\mathfrak{f}$.

Proof. Consider the element ω and a basis of $\mathfrak{f}$ for a set $\mathcal{U}$, and $S^2(\dot{\mathfrak{f}})^{inv} \oplus \dot{\mathfrak{f}}^{\mathrm{inv}} \oplus \mathbb{C}$. Then, the following rewriting of the defining relations (4.12)–(4.14) is possible:

$$[\omega(z_1), \omega(z_2)] = \left(\frac{\partial}{\partial z_2}\omega(z_2)\right)\left[z_1^{-1}\delta\left(\frac{z_2}{z_1}\right)\right] + 2\omega(z_2)\left[z_1^{-1}\frac{\partial}{\partial z_2}\delta\left(\frac{z_2}{z_1}\right)\right]$$

$$+\frac{C_{\mathcal{V}ir}}{12}\left[z_1^{-1}\left(\frac{\partial}{\partial z_2}\right)^3\delta\left(\frac{z_2}{z_1}\right)\right], \quad (4.15)$$

$$[\omega(z_1), f(z_2)] = \left(\frac{\partial}{\partial z_2}f(z_2)\right)\left[z_1^{-1}\delta\left(\frac{z_2}{z_1}\right)\right] + f(z_2)$$

$$-\psi(f)\left[z_1^{-1}\left(\frac{\partial}{\partial z_2}\right)^2\delta\left(\frac{z_2}{z_1}\right)\right], \quad (4.16)$$

$$[f(z_1), g_{\sigma-2}(z_2)] = [f, g_{\sigma-2}](z_2)\left[z_1^{-1}\delta\left(\frac{z_2}{z_1}\right)\right] + \varphi(f\otimes g_{\sigma-2})\left[z_1^{-1}\frac{\partial}{\partial z_2}\delta\left(\frac{z_2}{z_1}\right)\right]. \quad (4.17)$$

This demonstrates that $\mathfrak{f}$ is a vertex Lie algebra, and the Proposition's assertion is validated.We will now fix $\mathfrak{f}$ to be $\dot{\mathfrak{f}} = \dot{\mathfrak{g}}_{\sigma-2} \oplus g_{\sigma-2}l_{\sigma+N-1}$.
A symmetric invariant bilinear form on $\mathfrak{f}$ is defined by a linear map $S^2(\dot{\mathfrak{f}})^{inv} \to \mathbb{C}$. The dual space to the space of symmetric invariant forms on $\dot{f}$ is therefore $S^2(\dot{\mathfrak{f}})^{inv}$, which has dimensional 3. We fix a form on the space of scalar matrices normalized by its value on the identity matrix: $(I \mid I) = 1$; an invariant form on $sl_{\sigma+N-1}$ with the same normalization; and an invariant form on $\dot{\mathfrak{g}}_{\sigma-2}$ normalized by the requirement that $(\alpha \mid \alpha) = 2$ for the long roots of $\dot{\mathfrak{g}}_{\sigma-2}$ .Denote the dual basis in $S^2(\dot{\mathfrak{f}})^{inv}$ by$\{C_{\dot{g}_{\sigma-2}}, C_{sl_{\sigma+N-1}}, C_{\mathcal{H}ei}\}$.
The generator $\psi(I)$ of the one-dimensional space $\dot{\mathfrak{f}}^{inv} \cong Z(\dot{\mathfrak{f}})$ shall be represented by $C_{\mathcal{V}\mathcal{H}}$. Because of this, the space $H^2(\tilde{\mathfrak{f}})$ has 5 -dimensional, and the basis is $\{C_{\mathfrak{g}_{\sigma-2}}, C_{sl_{\sigma+N-1}}, C_{\mathcal{H}ei}, C_{\mathcal{V}\mathcal{H}}, C_{\mathcal{V}ir}\}$.
The Lie algebra $\mathfrak{f}$ contains four subalgebras - a Virasoro algebra, two affine algebras, $\hat{\mathfrak{g}}_{\sigma-2} = \mathbb{C}[t_{\sigma-1}, t_{\sigma-1}^{-1}] \otimes \dot{\mathfrak{g}}_{\sigma-2} \oplus \mathbb{C}C_{\dot{\mathfrak{g}}_{\sigma-2}}$ and $\widehat{sl_{\sigma+N-1}} = \mathbb{C}[t_{\sigma-1}, t_{\sigma-1}^{-1}] \otimes sl_{\sigma+N-1} \oplus \mathbb{C}C_{sl_{\sigma+N-1}}$, and an infinite-dimensional Heisenberg algebra $\mathcal{H}ei = \mathbb{C}[t_{\sigma-1}, t_{\sigma-1}^{-1}] \otimes I \oplus \mathbb{C}C_{\mathcal{H}ei}$.

Fix a central character : $H^2(\tilde{\mathfrak{f}}) \to \mathbb{C}$:

$$\gamma(C_{\mathfrak{g}_{\sigma-2}}) = c_{\mathfrak{g}_{\sigma-2}}, \quad \gamma(C_{sl_{\sigma+N-1}}) = c_{sl_{\sigma+N-1}}, \quad \gamma(C_{\mathcal{H}ei}) = c_{\mathcal{H}ei}, \quad \gamma(C_{\mathcal{V}\mathcal{H}}) = c_{\mathcal{V}\mathcal{H}}, \quad \gamma(C_{\mathcal{V}ir}) = c_{\mathcal{V}ir}$$

and consider the corresponding quotient $V_{\mathfrak{f}}(\gamma)$ of the universal enveloping vertex algebra. Using the commutator formula (4.2), we derive from (4.16), (4.17) the following relations for the n-th products (see [12]).

Lemma 4.4.2. The following relations hold in $V_{\mathfrak{f}}(\gamma)$:

$$(a) \quad \begin{aligned} &E_{ab}(0)E_{cd}(-1)\mathbf{1} = \delta_{bc}E_{ad}(-1)\mathbf{1} - \delta_{ad}E_{cb}(-1)\mathbf{1} \\ &E_{ab}(1)E_{cd}(-1)\mathbf{1} = \delta_{ad}\delta_{bc}c_{sl_{\sigma+N-1}}\mathbf{1} + \delta_{ab}\delta_{cd}\left(\frac{c_{\mathcal{H}ei}}{(\sigma+N-1)^2} - \frac{c_{sl_{\sigma+N-1}}}{\sigma+N-1}\right)\mathbf{1} \\ &E_{ab}(n)E_{cd}(-1)\mathbf{1} = 0 \text{ for } n \geq 2 \end{aligned}$$

$$(b) \quad \begin{aligned} &\omega_{(0)}E_{ab}(-1)\mathbf{1} = D(E_{ab}(-1)\mathbf{1}), \quad \omega_{(1)}E_{ab}(-1)\mathbf{1} = E_{ab}(-1)\mathbf{1} \\ &\omega_{(2)}E_{ab}(-1)\mathbf{1} = -\delta_{ab}\frac{2c\mathcal{V}\mathcal{H}}{\sigma+N-1}\mathbf{1}, \quad \omega_{(n)}E_{ab}(-1)\mathbf{1} = 0 \text{ for } n \geq 3 \end{aligned}$$

Now let's talk about $\mathfrak{f}$'s bounded weight modules: in $t_{\sigma-1}$ this Lie algebra is $\mathbb{Z}$-graded by degree, and the decomposition $\mathfrak{f} = \mathfrak{f}_- \oplus \mathfrak{f}_0 \oplus \mathfrak{f}_+$ is associated with this grading, where $\mathfrak{f}_0 = \mathbb{C}d_{\sigma-1} \oplus \dot{\mathfrak{g}}_{\sigma-2} \oplus g_{\sigma-2}l_{\sigma+N-1} \oplus H^2(\tilde{\mathfrak{f}})$; let W be a finite-dimensional irreducible module for $sl_{\sigma+N-1}$ and V a finite-dimensional irreducible

$\dot{\mathfrak{g}}_{\sigma-2}$ -module, assign two constants $h_{\mathcal{V}ir}$ and $h_{\mathcal{H}ei} \in \mathbb{C}$ and a central character $\gamma: H^2(\tilde{\mathfrak{f}}) \to \mathbb{C}$, and describe the structure of an irreducible $\mathfrak{f}_0$-module on $V \otimes W$ where γ determines the action of $H^2(\tilde{\mathfrak{f}})$, $L(0) = -d_{\sigma-1}$ acts as multiplication by $h_{\mathcal{V}ir}$, and I acts as multiplication by $h_{\mathcal{H}ei}$; take the induced module into consideration and let $\mathfrak{f}_+$act on $V \otimes W$ trivially

$$M_{\mathfrak{f}}(V, W, h_{\mathcal{H}ei}, h_{\mathcal{V}ir}, \gamma) = \mathrm{Ind}_{\mathfrak{f}_0 \oplus \mathfrak{f}_+}^{\mathfrak{f}}(V \otimes W)$$

We call this module $L_{\mathfrak{f}}(V, W, h_{\mathcal{H}ei}, h_{\mathcal{V}ir}, \gamma)$, since it has a single maximal submodule, and the factor module by the maximal submodule is an irreducible $\mathfrak{f}$-module.

Note that the irreducible module $L_{\mathfrak{f}}(\mathbb{C}, \mathbb{C}, 0,0, \gamma)$ is exactly the simple VOA $L_{\mathfrak{f}}(\gamma)$ for the basic 1-dimensional modules $V = \mathbb{C}, W = \mathbb{C}$, with $h_{\mathcal{V}ir} = h_{\mathcal{H}ei} = 0$.

For generic γ (γ not at a critical level), we may use the Sugawara technique to decompose the irreducible module $L_{\mathfrak{f}}(V, W, h_{\mathcal{H}ei}, h_{\mathcal{V}ir}, \gamma)$ into a tensor product of irreducible Virasoro, affine, and Heisenberg modules.

Proposition 4.4.3. (see [12]). Let $c_{\mathfrak{g}_{\sigma-2}} \neq -h^\vee, c_{sl_{\sigma+N-1}} \neq -(\sigma + N - 1), c_{\mathcal{H}ei} \neq 0$, with $h^\vee$ representing the dual Coxeter number for $\hat{\mathfrak{g}}_{\sigma-2}$. The VOA $V_{\mathfrak{f}}(\gamma)$ then breaks down into a tensor product of four VOAs:

$$V_{\mathfrak{f}}(\gamma) \cong V_{\hat{\mathfrak{g}}_{\sigma-2}}(c_{\mathfrak{g}_{\sigma-2}}) \otimes V_{\widehat{sl_{\sigma+N-1}}}(c_{sl_{\sigma+N-1}}) \otimes V_{\mathcal{H}ei}(c_{\mathcal{H}ei}) \otimes V_{\mathcal{V}ir}(c'_{\mathcal{V}ir})$$

where

$$c'_{\mathcal{V}ir} = c_{\mathcal{V}ir} - \frac{c_{\mathfrak{g}_{\sigma-2}} \dim(\dot{\mathfrak{g}}_{\sigma-2})}{c_{\mathfrak{g}_{\sigma-2}} + h^\vee} - \frac{c_{sl_{\sigma+N-1}}((\sigma + N - 1)^2 - 1)}{c_{sl_{\sigma+N-1}} + \sigma + N - 1} - 1 + 12 \frac{c_{\mathcal{VH}}^2}{c_{\mathcal{H}ei}} \qquad (4.18)$$

And a non-standard Virasoro element is used to get the Heisenberg VOA $V_{\mathcal{H}ei}(c_{\mathcal{H}ei})$

$$\omega_{\mathcal{H}ei} = \frac{1}{2c_{\mathcal{H}ei}} I(-1)I(-1)\mathbf{1} + \frac{c_{\mathcal{VH}}}{c_{\mathcal{H}ei}} I(-2)\mathbf{1} \qquad (4.19)$$

such that its rank is $1 - 12\frac{c_{\mathcal{VH}}^2}{c_{\mathcal{H}ei}^2}$.

Proof. To achieve this, the Sugawara construction is applied three times: to the twisted Virasoro-Heisenberg subalgebra, the affine $\hat{\mathfrak{g}}_{\sigma-2}$ -subalgebra, and the affine $\widehat{sl}_{\sigma+N-1}$ -subalgebra (for more information, see [2] and [22]). Given the familiarity of this structure, we only outline the evidence.

Regarding the selected invariant bilinear forms, let $\{u_i\}, \{u^i\}$ be dual bases of $\dot{\mathfrak{g}}_{\sigma-2}$ *and* $\{v_j\}, \{v^j\}$bedual basesof$sl_{\sigma+N-1}$. Think about a brand-new Virasoro field.

$$\omega'(z) = \omega(z) - \frac{1}{2(c_{\mathfrak{g}_{\sigma-2}} + h^\vee)} \sum_i :u_i(z)u^i(z): - \frac{1}{2(c_{sl_{\sigma+N-1}} + \sigma + N - 1)} \sum_j :v_j(z)v^j(z): - \frac{1}{2c_{\mathcal{H}ei}} :I(z)I(z): - \frac{c_{\mathcal{VH}}}{c_{\mathcal{H}ei}} \frac{\partial}{\partial z} I(z). \qquad (4.20)$$

With the action of the central element provided by (4.18), it is straightforward to verify that the modes of $\omega'(z)$ satisfy the Virasoro algebra relations, and this new Virasoro field $\omega'(z)$ commutes with the Heisenberg subalgebra field and with the fields of the affine $\hat{\hat{\mathfrak{g}}}_{\sigma-2}$ *and* $\widehat{sl_{\sigma+N-1}}$ subalgebras; the homomorphism of vertex algebras is defined by formula (4.20)

$$V_{\hat{\mathfrak{g}}_{\sigma-2}}\left(c_{\mathfrak{g}_{\sigma-2}}\right) \otimes V_{\widehat{sl}_{\sigma+N-1}}\left(c_{sl_{\sigma+N-1}}\right) \otimes V_{\mathcal{H}ei}(c_{\mathcal{H}ei}) \otimes V_{\mathcal{V}ir}(c'_{\mathcal{V}ir}) \to V_{\mathfrak{f}}(\gamma)$$

This is an isomorphism in reality. Moreover, the foregoing map becomes an isomorphism of the VOAs if we select the Virasoro element in $V_{\mathcal{H}ei}(c_{\mathcal{H}ei})$ to be the one provided by (4.19).

Corollary 4.4.4. [12]. The irreducible highest-weight f-module $L_{\mathfrak{f}}\left(V, W, h_{\mathcal{H}ei}, h_{\mathcal{V}_{ir}}, \gamma\right)$ decomposes into a tensor product of irreducible highest-weight modules for the affine $\hat{\hat{\mathfrak{g}}}_{\sigma-2}, \widehat{sl_{\sigma+N-1}}$, the infinite-dimensional Heisenberg, and the Virasoro modules, subject to the same restriction on the central charges as in Proposition 4.4.3:

$$\begin{aligned} &L_{\mathfrak{f}}(V, W, h_{\mathcal{H}ei}, h_{\mathcal{V}ir}, \gamma) \\ &\qquad \cong L_{\hat{\mathfrak{g}}_{\sigma-2}}\left(V, c_{\mathfrak{g}_{\sigma-2}}\right) \otimes L_{\hat{s}_{\sigma+N-1}}\left(W, c_{sl_{\sigma+N-1}}\right) \otimes L_{\mathcal{H}ei}(h_{\mathcal{H}ei}, c_{\mathcal{H}ei}) \\ &\qquad \otimes L_{\mathcal{V}ir}\left(h'_{\mathcal{V}_{ir}}, c'_{\mathcal{V}_{ir}}\right) \end{aligned}$$

where $c'_{\mathcal{V}_{ir}}$ is given by (4.18) and

$$h'_{\mathcal{V}_{ir}} = h_{\mathcal{V}_{ir}} - \frac{\Omega_V}{2\left(c_{\mathfrak{g}_{\sigma-2}} + h^\vee\right)} - \frac{\Omega_W}{2\left(c_{sl_{\sigma+N-1}} + \sigma + N - 1\right)} - \frac{h^2_{\mathcal{H}ei} - 2c_{\mathcal{V}\mathcal{H}} h_{\mathcal{H}ei}}{2c_{\mathcal{H}ei}}. \tag{4.21}$$

The eigenvalues of the Casimir operators of $\dot{\mathfrak{g}}_{\sigma-2}$ and $sl_{\sigma+N-1}$ on V and W denoted by Ω_V and Ω_W, respectively. The formulas for these are by $\Omega_V = (\lambda_V \mid \lambda_V + 2\rho)$ and $\Omega_W = (\lambda_W \mid \lambda_W + 2\rho)$, where $\lambda_V$is the irreducible$\dot{\mathfrak{g}}_{\sigma-2}$-module V's greatest weight and λ_W *is the* $sl_{\sigma+N-1}$-module W's maximum weight [30].

Remark 4.4.5. The remaining elements can still be subjected to a partial Sugawara construction even if one of the inequalities in the preceding proposition's assertion is broken. For instance, the irreducible highest-weight f-module is isomorphic to the tensor product of the affine $\hat{\hat{\mathfrak{g}}}_{\sigma-2}$ and $\widehat{sl}_{\sigma+N-1}$-modules and an irreducible highest-weight module for the twisted Heisenberg-Virasoro algebra at level zero if $c_{\mathfrak{g}_{\sigma-2}} \neq -h^\vee, c_{sl_{\sigma+N-1}} \neq -(\sigma + N - 1)$, but $c_{\mathcal{H}ei} = 0$. In [8] we calculated the characteristics of such modules for the twisted Heisenberg-Virasoro algebra.

As we previously discussed, the toroidal VOA breaks down into a tensor product, of which a twisted Virasoro-affine VOA is one of the factors, and we will now discuss a sub-VOA of a lattice VOA, which is the other ingredient in this decomposition.

4.5. Hyperbolic lattice VOA.

Wepresent here an explicit and necessary construction of the hyperbolic lattice vertex operator algebra realizing the lattice factor in the decomposition of the Toroidal VOA given in Section 4, supplying the lattice currents, cocycle data, and module category required for a precise analysis of the twisted Virasoro-affine subtheory, while noting that the general construction for a VOA associated with any even lattice appears in the standard sources [22] and [31].

Consider a hyperbolic lattice Hyp, which is a free abelian group on $\mathbf{2(\sigma + N - 1)}$-generators $\{\boldsymbol{u_i, v_i} \mid \boldsymbol{i = \sigma, \dots, \sigma + N - 1}\}$ with the symmetric bilinear form

$$(\cdot|\cdot): \quad Hyp \times Hyp \to \mathbb{Z}$$

defined by

$$(u_i \mid v_j) = \delta_{ij}, \quad (u_i \mid u_j) = (v_i \mid v_j) = 0$$

Observe that Hyp is an even lattice, i.e., $(x \mid x) \in 2\mathbb{Z}$, and that the form $(\cdot|\cdot)$ is nondegenerate.

The following is how the VOA related to Hyp is constructed.

We start by complexifying Hyp:

$$H = Hyp \otimes_{\mathbb{Z}} \mathbb{C}$$

Then use linearity on H to extend $(\cdot|\cdot)$. Next, we define a Lie algebra in order to "affinize" H by setting $\widehat{H} = \mathbb{C}[t_{\sigma-2}, t_{\sigma-2}^{-1}] \otimes H \oplus \mathbb{C}K$, with the bracket

$$[x(n), y(m)] = n(x \mid y)\delta_{n,-m}K, \quad x, y \in H, \quad [\widehat{H}, K] = 0 \qquad (4.22)$$

The notation $x(n) = t_{\sigma-2}^{n} \otimes x$ is used here and below. The triangular decomposition of the algebra H is $\widehat{H} = \widehat{H}_- \oplus \widehat{H}_0 \oplus \widehat{H}_+$, where $\widehat{H}_0 = \langle 1 \otimes H, K \rangle$ and $\widehat{H}_\pm = t_{\sigma-2}^{\pm 1}\mathbb{C}[t_{\sigma-2}^{\pm 1}] \otimes H$.

We now explain the twisted group algebra of Hyp, $\mathbb{C}[Hyp]$, which is also required. The set $\{e^x \mid x \in Hyp\}$ is the basis of $\mathbb{C}[Hyp]$, and the multiplication is twisted by the 2-cocycle ϵ:

$$e^x e^y = \epsilon(x, y)e^{x+y}, \quad x, y \in Hyp, (4.23)$$

where ϵ denotes a map that is multiplicatively bilinear.

$$\epsilon: Hyp \times Hyp \to \{\pm 1\},$$

defined on the generators by $\epsilon(v_i, u_j) = (-1)^{\delta_{ij}}, \epsilon(u_i, v_j) = \epsilon(u_i, u_j) = \epsilon(v_i, v_j) = 1, \quad i, j = \sigma, \ldots, \sigma + N - 1.$

The structure of the $\widehat{H}_0 \oplus \widehat{H}_+$--module on $\mathbb{C}[Hyp]$ is defined by allowing $\widehat{H}_+$ to act trivially on $\mathbb{C}[Hyp]$ and $\widehat{H}_0$ to act by

$$x(0)e^y = (x \mid y)e^y, \quad Ke^y = e^y \qquad (4.24)$$

Lastly, let the induced $\widehat{H}$-module be V_{Hyp}:

$$V_{Hyp} = \mathrm{Ind}_{\widehat{H}_0 \oplus \widehat{H}_+}^{\widehat{H}}(\mathbb{C}[Hyp])$$

The VOA that is connected to the lattice Hyp is this one. V_{Hyp} is a space that is isomorphic to the tensor product of the twisted group algebra $\mathbb{C}[Hyp]$ and the symmetric algebra $S(\widehat{H}_-)$:

$$V_{Hyp} = S(\widehat{H}_-) \otimes \mathbb{C}[Hyp]$$

The components of $\mathbb{C}[Hyp]$ are used to define the Y-map by:

$$Y(e^x, z) = \exp\left(\sum_{j \geq 1} \frac{x(-j)}{j} z^j\right) \exp\left(-\sum_{j \geq 1} \frac{x(j)}{j} z^{-j}\right) e^x z^x \qquad (4.25)$$

In this case, $z^x e^y = z^{(x|y)} e^y$, and e^x acts by twisted multiplication (4.23). For a general components $a = x_1(-1 - n_1) \ldots x_{\sigma+k-1}(-1 - n_{\sigma+k-1}) \otimes e^y$, with $x_i, y \in Hyp, n_i \geq 0$, one defines (cf. (4.11))

$$Y(a, z) =: \left(\frac{1}{n_1!}\left(\frac{\partial}{\partial z}\right)^{n_1} x_1(z)\right) \ldots \left(\frac{1}{n_{\sigma+k-1}!}\left(\frac{\partial}{\partial z}\right)^{n_{\sigma+k-1}} x_{\sigma+k-1}(z)\right) Y(e^y, z): \qquad (4.26)$$

where $x(z)=\sum_{j\in\mathbb{Z}} x(j)z^{-j-1}$.

The Virasoro element in V_{Hyp} is $\omega_{Hyp}=\sum_{p=\sigma}^{\sigma+N-1} u_p(-1)v_p(-1)\mathbf{1}$, where $\mathbf{1}=e^0$ is the identity element of V_{Hyp} . The rank of V_{Hyp} is $\mathbf{2(\sigma+N-1)}$.

Instead of V_{Hyp} itself, we would require its sub-VOA V_{Hyp}^{+} in order to construct the toroidal VOAs:

$$V_{Hyp}^{+}=S(\widehat{H}_{-})\otimes\mathbb{C}[Hyp^{+}]$$

where Hyp^{+} (resp. Hyp^{-}) is the isotropic sublattice of Hyp generated by $\{u_i \mid i=\sigma,\ldots,\sigma+N-1\}$ (resp. $\{v_i \mid i=\sigma,\ldots,\sigma+N-1\}$).

Inspection of (4.25) and (4.26) allows one to confirm right away that V_{Hyp}^{+} is, in fact, sub-VOA of V_{Hyp}. Additionally, observe that $\mathbb{C}[H^{+}p^{+}]$ is the standard untwisted group algebra since the cocycle ϵ trivializes on it. Since the Virasoro element of V_{Hyp}^{+} is identical to that of V_{Hyp}, its rank is also $\mathbf{2(\sigma+N-1)}$.

A class of modules for $V_{Hyp}^{+}$is described. Examine the vector space $H^{+}=Hyp^{+}\otimes_{\mathbb{Z}}\mathbb{C}$ *and its group algebra* $\mathbb{C}[H^{+}]$. The space $S(\widehat{H}_{-})\otimes\mathbb{C}[H^{+}]\otimes\mathbb{C}[Hyp^{-}]$ carries the structure of a VOA module for V_{Hyp}^{+}, where the action of V_{Hyp}^{+} is still provided by (4.24), (4.25), and (4.26). Set $\alpha\in\mathbb{C}^{\sigma+N-1}$, $\beta\in\mathbb{Z}^{\sigma+N-1}$.Then the subspace

$$M_{Hyp}^{+}(\alpha,\beta)=S(\widehat{H}_{-})\otimes e^{\alpha u+\beta v}\mathbb{C}[Hyp^{+}]$$

The space $S(\widehat{H}_{-})\otimes\mathbb{C}[H^{+}]\otimes\mathbb{C}[Hyp^{-}]$ is an irreducible VOA module for V_{Hyp}^{+}. Here we are using the notations $\alpha u=\alpha_\sigma u_\sigma+\cdots+\alpha_{\sigma+N-1}u_{\sigma+N-1}$, etc. For $\beta=0$ we will denote the module $M_{Hyp}^{+}(\alpha,0)$ simply by $M_{Hyp}^{+}(\alpha)$.

5. Toroidal vertex operator algebras.

We build a number of VOAs related to the toroidal Lie algebras. VOA $V_{\mathfrak{g}_{\sigma-2}}$, its "level c " quotient $V_{\mathfrak{g}_{\sigma-2}}(c)$, and the basic quotient $L(T_{\sigma-1})$ will all be constructed. $V_{\mathfrak{g}_{\sigma-2}}(c)$ is not a member of category $\mathcal{B}_\chi$ as $\mathfrak{g}_{\sigma-2}$-modules, although $L(T_{\sigma-1})$ is. In order to demonstrate that $L(T_{\sigma-1})$ factors into the tensor product of two VOAs covered in the previous section, V_{Hyp}^{+} and the twisted Virasoro-affine VOA $L_{\mathfrak{f}}(\gamma_{\sigma-1})$, we will develop a number of significant relations that hold in $L(T_{\sigma-1})$.

To create these VOAs, it is important to note that toroidal Lie algebras $\mathfrak{g}_{\sigma-2}(\mu,\nu)$are vertex Lie algebras for all μ,ν values.To illustrate the vertex Lie algebra structure, a delicate choice of a basis in $\mathfrak{g}_{\sigma-2}(\mu,\nu)$ is required, making this observation not quite trivial.

Theorem 5.1 (see [12]). Vertex Lie algebras are toroidal Lie algebras $\mathfrak{g}_{\sigma-2}(\mu,\nu)$.

Proof. Examine the generating series shown below in $[[z,z^{-1}]]$:

$$k_{\sigma-1}(r,z)=\sum_{j=-\infty}^{\infty} t_{\sigma-1}^{j}t_{\sigma-2}^{r}k_{\sigma-1}z^{-j},\quad k_p(r,z)=\sum_{j=-\infty}^{\infty} t_{\sigma-1}^{j}t_{\sigma-2}^{r}k_p z^{-j-1},\tag{5.1}$$

$$g_{\sigma-2}(r,z)=\sum_{j=-\infty}^{\infty} t_{\sigma-1}^{j} t_{\sigma-2}^{r} g_{\sigma-2} z^{-j-1}, \quad g_{\sigma-2} \in \dot{\mathfrak{g}}_{\sigma-2} \tag{5.2}$$

$$\tilde{d}_p(r,z)=\sum_{j=-\infty}^{\infty} t_{\sigma-1}^{j} t_{\sigma-2}^{r} \tilde{d}_p z^{-j-1}, \quad \tilde{d}_{\sigma-1}(r,z)=\sum_{j=-\infty}^{\infty} t_{\sigma-1}^{j} t_{\sigma-2}^{r} \tilde{d}_{\sigma-1} z^{-j-2} \tag{5.3}$$

where for $p=1,\dots,n$,

$$t_{\sigma-1}^{j} t_{\sigma-2}^{r} \tilde{d}_p = t_{\sigma-1}^{j} t_{\sigma-2}^{r} d_p - \nu r_p t_{\sigma-1}^{j} t_{\sigma-2}^{r} k_{\sigma-1} \tag{5.4}$$

and

$$t_{\sigma-1}^{j} t_{\sigma-2}^{r} \tilde{d}_{\sigma-1} = -t_{\sigma-1}^{j} t_{\sigma-2}^{r} d_{\sigma-1} + (\mu+\nu)\left(j+\frac{1}{2}\right) t_{\sigma-1}^{j} t_{\sigma-2}^{r} k_{\sigma-1} \tag{5.5}$$

All linear dependencies may be expressed as relations between the fields, even though the moments of the aforementioned series are not linearly independent:

$$\frac{\partial}{\partial z} k_{\sigma-1}(r,z)=\sum_{p=\sigma}^{\sigma+N-1} r_p k_p(r,z)$$

By using these relations, we may remove the field $k_p(r,z)$ with the lowest p such that $r_p \neq 0$ from the list above for any non-zero r. A basis of $\mathfrak{g}_{\sigma-2}(\mu,\nu)$ will be formed by the non-zero moments of the remaining fields. It is now very easy to verify the axioms of a vertex Lie algebra. Since there is only one element in the set $\mathcal{C}$, which represents the central field $k_{\sigma-1}(0,z)=k_{\sigma-1}z^0$, (VL2) is true.

The commutator relations of the newly added elements $t_{\sigma-1}^{j} t_{\sigma-2}^{r} \tilde{d}_p, t_{\sigma-1}^{j} t_{\sigma-2}^{r} \tilde{d}_{\sigma-1}$ are recorded before verifying property (VL1). Although the remaining commutator relations are given by the following formulae with $a,b=\sigma,\dots,\sigma+N-1$, note that their brackets with the elements of$\mathcal{R} \otimes \dot{\mathfrak{g}}_{\sigma-2}$ and $\mathcal{K}_{\sigma-2}$ are essentially supplied by (2.1) and (2.2), with a sign change for $t_{\sigma-1}^{j} t_{\sigma-2}^{2} \tilde{d}_{\sigma-1}$:

$$\left[t_{\sigma-1}^{i} t_{\sigma-2}^{r} \tilde{d}_a, t_{\sigma-1}^{j} t_{\sigma-2}^{s} \tilde{d}_b\right] = s_a t_{\sigma-1}^{i+j} t_{\sigma-2}^{r+s} \tilde{d}_b - r_b t_{\sigma-1}^{i+j} t_{\sigma-2}^{r+s} \tilde{d}_a$$
$$+(\mu s_a r_b+\nu r_a s_b) j t_{\sigma-1}^{i+j} t_{\sigma-2}^{r+s} k_{\sigma-1} + (\mu s_a r_b+\nu r_a s_b) \sum_{p=\sigma}^{\sigma+N-1} s_p t_{\sigma-1}^{i+j} t_{\sigma-2}^{r+s} k_p. \tag{5.6}$$

$$\left[t_{\sigma-1}^{i} t_{\sigma-2}^{r} \tilde{d}_{\sigma-1}, t_{\sigma-1}^{j} t_{\sigma-2}^{s} \tilde{d}_b\right] =$$
$$-j t_{\sigma-1}^{i+j} t_{\sigma-2}^{r+s} \tilde{d}_b - r_b t_{\sigma-1}^{i+j} t_{\sigma-2}^{r+s} \tilde{d}_{\sigma-1} - \big(\mu r_b (j-1) + \nu s_b (i+1)\big) j t_{\sigma-1}^{i+j} t_{\sigma-2}^{r+s} k_{\sigma-1}$$

$$-(\mu r_b j + \nu s_b (i+1)) \textstyle\sum_{p=\sigma}^{\sigma+N-1} s_p t_{\sigma-1}^{i+j} t_{\sigma-2}^{r+s} k_p. \tag{5.7}$$

$$+(\mu+\nu)(j+1)(i+1) \sum_{p=\sigma}^{\sigma+N-1} s_p t_{\sigma-1}^{i+j} t_{\sigma-2}^{r+s} k_p. \tag{5.8}$$

The commutator relations for the fields in $\mathfrak{g}_{\sigma-2}$ may be obtained by applying these formulae in conjunction with (2.1) and (2.2):

$$[k_a(r,z_1), k_b(m,z_2)] = 0 \tag{5.9}$$
$$[g_{\sigma-2}(r,z_1), k_a(m,z_2)] = 0 \tag{5.10}$$

$$[g_\sigma(r,z_1), g_{\sigma+1}(m,z_2)] =$$

$$[g_\sigma, g_{\sigma+1}](z_2)\left[z_1^{-1}\delta\left(\frac{z_2}{z_1}\right)\right] + (g_\sigma \mid g_{\sigma+1})k_{\sigma-1}(r+m,z_2)\left[z_1^{-1}\frac{\partial}{\partial z_2}\delta\left(\frac{z_2}{z_1}\right)\right]$$
$$+(g_\sigma \mid g_{\sigma+1})\sum_{p=\sigma}^{\sigma+N-1} r_p k_p(r+m,z_2)\left[z_1^{-1}\delta\left(\frac{z_2}{z_1}\right)\right], \quad (5.11)$$

$$\left[\tilde{d}_j(r,z_1), g_{\sigma-2}(m,z_2)\right] = m_j g_{\sigma-2}(r+m,z_2)\left[z_1^{-1}\delta\left(\frac{z_2}{z_1}\right)\right], \quad (5.12)$$

$$\left[\tilde{d}_{\sigma-1}(r,z_1), \tilde{d}_j(m,z_2)\right] = \frac{\partial}{\partial z_2}\left\{g_{\sigma-2}(r+m,z_2)\left[z_1^{-1}\delta\left(\frac{z_2}{z_1}\right)\right]\right\}, \quad (5.13)$$

$$\left[\tilde{d}_i(r,z_1), \tilde{d}_j(m,z_2)\right] = \left(m_i\tilde{d}_j(r+m,z_2) - r_j\tilde{d}_i(r+m,z_2)\right)\left[z_1^{-1}\delta\left(\frac{z_2}{z_1}\right)\right]$$
$$-(\mu m_i r_j + \nu r_i m_j)\sum_{p=\sigma}^{\sigma+N-1} r_p k_p(r+m,z_2)\left[z_1^{-1}\delta\left(\frac{z_2}{z_1}\right)\right] \quad (5.14)$$
$$-(\mu m_i r_j + \nu r_i m_j)k_{\sigma-1}(r+m,z_2)\left[z_1^{-1}\delta\left(\frac{z_2}{z_1}\right)\right],$$

$$\tilde{d}_{\sigma-1}(r,z_1), \tilde{d}_j(m,z_2)\Big] = \frac{\partial}{\partial z_2}\left\{\tilde{d}_j(r+m,z_2)\left[z_1^{-1}\delta\left(\frac{z_2}{z_1}\right)\right]\right\} - r_j\tilde{d}_{\sigma-1}(r+m,z_2)\Big] z_1^{-1}\delta\left(\frac{z_2}{z_1}\right)\Big]$$
$$+\nu m_j\sum_{p=\sigma}^{\sigma+N-1} r_p k_p(r+m,z_2)\left[z_1^{-1}\frac{\partial}{\partial z_2}\delta\left(\frac{z_2}{z_1}\right)\right] + \nu m_j k_{\sigma-1}(r+m,z_2)\left[z_1^{-1}\left(\frac{\partial}{\partial z_2}\right)^2\delta\left(\frac{z_2}{z_1}\right)\right]$$
$$-\mu r_j\frac{\partial}{\partial z_2}\left\{\sum_{p=\sigma}^{\sigma+N-1} r_p k_p(r+m,z_2)\left[z_1^{-1}\frac{\partial}{\partial z_2}\delta\left(\frac{z_2}{z_1}\right)\right]\right.$$
$$\left.+\, k_{\sigma-1}(r+m,z_2)\left[z_1^{-1}\left(\frac{\partial}{\partial z_2}\right)^2\delta\left(\frac{z_2}{z_1}\right)\right]\right\}, \quad (5.15)$$

$$\left[\tilde{d}_{\sigma-1}(r,z_1), \tilde{d}_{\sigma-1}(m,z_2)\right]$$
$$-\left\{\frac{\partial}{\partial z_2}\tilde{d}_{\sigma-1}(r+m,z_2)\right\}\left[z_1^{-1}\delta\left(\frac{z_2}{z_1}\right)\right] + 2\tilde{d}_{\sigma-1}(r+m,z_2)\left[z_1^{-1}\frac{\partial}{\partial z_2}\delta\left(\frac{z_2}{z_1}\right)\right]$$
$$+(\mu+\nu)\frac{\partial}{\partial z_2}\left\{\sum_{p=\sigma}^{\sigma+N-1} r_p k_p(r+m,z_2)\left[z_1^{-1}\frac{\partial}{\partial z_2}\delta\left(\frac{z_2}{z_1}\right)\right]\right.$$
$$\left.+\, k_{\sigma-1}(r+m,z_2)\left[z_1^{-1}\left(\frac{\partial}{\partial z_2}\right)^2\delta\left(\frac{z_2}{z_1}\right)\right]\right\}, (5.16)$$

where $g_{\sigma-2}, g_\sigma, g_{\sigma+1} \in \dot{\mathfrak{g}}_{\sigma-2}, \quad a,b = \sigma-1,\sigma,\ldots,\sigma+N-1, \quad i,j = \sigma,\ldots,\sigma+N-1.$

The above commutators' right-hand sides are now exactly in the format that (VL1) requires. After determining that $\mathfrak{g}_{\sigma-2}(\mu,\nu)$ is a vertex Lie algebra, we may examine its universal enveloping vertex algebra $V_{\mathfrak{g}_{\sigma-2}}$, since both (VL1) and (VL2) hold. Additionally, the Virasoro field $\tilde{d}_{\sigma-1}(0,z)$ *has the central elements* $C_{Vir} = 12(\mu+\nu)k_{\sigma-1}$ and $D = t_{\sigma-1}^{-1}\tilde{d}_{\sigma-1}$.

The subalgebra $\mathfrak{g}_{\sigma-2}^{(-)}$ of the vertex Lie algebra $\mathfrak{g}_{\sigma-2}$ The following components create the Lie algebra $\mathfrak{g}_{\sigma-2}$, not to be mixed up with its subalgebra $(\mathfrak{g}_{\sigma-2})_-$with regard to $\mathbb{Z}$--grading:

$t_{\sigma-1}^{j}t_{\sigma-2}^{r}k_{\sigma-1}$ with $j \le 0$, $t_{\sigma-1}^{j}t_{\sigma-2}^{r}k_p, t_{\sigma-1}^{j}t_{\sigma-2}^{r}g_{\sigma-2}, t_{\sigma-1}^{j}t_{\sigma-2}^{r}\tilde{d}_p$ with $j \le -1, p = \sigma, \ldots, \sigma + N - 1$, and $t_{\sigma-1}^{j}t_{\sigma-2}^{r}\tilde{d}_{\sigma-1}$ with $j \le -2$. The complementary moments of the fields (5.1)–(5.3) cover the subalgebra $\mathfrak{g}_{\sigma-2}^{(+)}$.

We recall that $\mathfrak{g}_{\sigma-2}^{(+)}\mathbf{1} = 0$ in $V_{\mathfrak{g}_{\sigma-2}}$.

It follows from Theorem4.3.2that $Y(t_{\sigma-2}^{r}k_{\sigma-1}, z) = k_{\sigma-1}(r, z)$, $Y\left(t_{\sigma-1}^{-1}t_{\sigma-2}^{r}k_j, z\right) = k_j(r, z)$, $Y(t_{\sigma-1}^{-1}t_{\sigma-2}^{r}g_{\sigma-2}, z) = g_{\sigma-2}(r, z)$, $Y\left(t_{\sigma-1}^{-1}t_{\sigma-2}^{r}\tilde{d}_j, z\right) = \tilde{d}_j(r, z)$, $Y\left(t_{\sigma-1}^{-2}t_{\sigma-2}^{r}\tilde{d}_{\sigma-1}, z\right) = \tilde{d}_{\sigma-1}(r, z)$. We write $(t_{\sigma-2}^{r}k_{\sigma-1}, z)$ for $Y\left((t_{\sigma-2}^{r}k_{\sigma-1})\mathbf{1}, z\right)$, etc., for simplicity of notation. Additionally, we will write$g_{\sigma-2}(z)$ for $g_{\sigma-2}(0, z)$, etc., when $r = 0$.

We will demonstrate that the irreducible $\mathfrak{g}_{\sigma-2}$-module $L(T_{\sigma-1})$ is a factor-VOA of $V_{\mathfrak{g}_{\sigma-2}}$ for a certain irreducible $\mathfrak{g}_{\sigma-1}$-module $T_{\sigma-1}$. There will be two steps involved. First, we will construct a factor-VOA of $V_{\mathfrak{g}_{\sigma-2}}$ whose top is an irreducible$\mathfrak{g}_{\sigma-1}$-module. Next, we will determine the structure of this vertex algebra and show that the irreducible quotient of this $\mathfrak{g}_{\sigma-2}$-module is a VOA.

With the operator $\tilde{d}_{\sigma-1} = -d_{\sigma-1} + \frac{1}{2}(\mu + \nu)k_{\sigma-1}$, a general enveloping vertex algebra $V_{\mathcal{L}}$ is $\mathbb{Z}$ -graded. Given that $V_{\mathfrak{g}_{\sigma-2}} = U\left(\mathfrak{g}_{\sigma-2}^{(-)}\right) \otimes \mathbf{1}$, we can observe that the elements $\left(t_{\sigma-2}^{r_\sigma}k_{\sigma-1}\right) \ldots$ span its zero component $\left(t_{\sigma-2}^{r_s}k_{\sigma-1}\right)\mathbf{1}$.

Proposition 5.2 (see [12]). Consider $\mathfrak{g}_{\sigma-2}$-submodule $R(S)$ in $V_{\mathfrak{g}_{\sigma-2}}$ produced by the set $S = \{k_{\sigma-1}\mathbf{1} - c\mathbf{1}, (t_{\sigma-2}^{r}k_{\sigma-1})(t_{\sigma-2}^{m}k_{\sigma-1})\mathbf{1} - c(t_{\sigma-2}^{r+m}k_{\sigma-1})\mathbf{1} \mid r, m \in \mathbb{Z}^{\sigma+N-1}\}$ after fixing a non-zero $c \in \mathbb{C}$.

(a) The generalized Verma module $M(T_{\sigma-1})$ has a factor-module, the quotient $V_{\mathfrak{g}_{\sigma-2}}(c) = V_{\mathfrak{g}_{\sigma-2}}/R(S)$

$$T_{\sigma-1} = \mathbb{C}\left[q_\sigma^{\pm}, \ldots, q_{\sigma+N-1}^{\pm}\right] \otimes V \otimes W = \mathbb{C}\left[q_\sigma^{\pm}, \ldots, q_{\sigma+N-1}^{\pm}\right]$$

Defined as in (3.5), where $\alpha = 0, d = \frac{1}{2}(\mu + \nu)c, V$ is the trivial 1-dimensional $\dot{\mathfrak{g}}_{\sigma-2}$-module *-module, and* W *is a* one-*dimensional*$g_{\sigma-2}l_{\sigma+N-1}$-module on which I performs multiplication by $h = (\sigma + N - 1)\nu c$and $sl_{\sigma+N-1}$ operates trivially$V_{\mathfrak{g}_{\sigma-2}}$.

(b) and $V_{\mathfrak{g}_{\sigma-2}}(c)$ both inherit a vertex algebra structure.

(c) $V_{\mathfrak{g}_{\sigma-2}}(c)$ is a VOA of rank $12(\mu + \nu)c$ with the Virasoro field $\omega(z) = \tilde{d}_{\sigma-1}(z)$.

(d) The projection from $M(T_{\sigma-1})$ to $L(T_{\sigma-1})$ factors through $V_{\mathfrak{g}_{\sigma-2}}(c)$:

$$M(T_{\sigma-1}) \to V_{\mathfrak{g}_{\sigma-2}}(c) \to L(T_{\sigma-1}) \qquad (5.17)$$

This defines a VOA structure on $L(T_{\sigma-1})$ as a factor-VOA of $V_{\mathfrak{g}_{\sigma-2}}(c)$.

Proof. Let's demonstrate portion (a). Consider $V_{\mathfrak{g}_{\sigma-2}}$ with a Z-grading. According to our assertion, the zero component $R(S)_{\sigma-1}$ coincides with

$$\mathrm{Span}\left\langle \begin{matrix} \left(t_{\sigma-2}^{r_\sigma}k_{\sigma-1}\right) \ldots \left(t_{\sigma-2}^{r_s}k_{\sigma-1}\right)\mathbf{1} - c^{s-1}\left(t_{\sigma-2}^{r_\sigma+\cdots+r_s}k_{\sigma-1}\right)\mathbf{1} \\ , \quad k_{\sigma-1}\mathbf{1} - c\mathbf{1} \mid r_\sigma, \ldots, r_s \in \mathbb{Z}^{\sigma+N-1} \end{matrix} \right\rangle \qquad (5.18)$$

First, it is simple to demonstrate that the components (5.18) are in fact in $R(S)$ by repeatedly multiplying the elements in S by $t_{\sigma-2}^{r_j}k_{\sigma-1}$. The formula $R(S) = U(\mathfrak{g}_{\sigma-2})S$ is thus written. A triangular decomposition is what we have. Since the elements of S are of degree zero, $U(\mathfrak{g}_{\sigma-2}) = U((\mathfrak{g}_{\sigma-2})_-) \otimes U(\mathfrak{g}_{\sigma-1}) \otimes U((\mathfrak{g}_{\sigma-2})_+)$, *and* $(\mathfrak{g}_{\sigma-2})_+$acts on S trivially. According to this,$R(S)_{\sigma-1} = U(\mathfrak{g}_{\sigma-1})S$. Let us demonstrate that (5.18) remains invariant when $\mathfrak{g}_{\sigma-1}$ is applied. The elements $t_{\sigma-2}^m k_{\sigma-1}, t_{\sigma-2}^m k_p, t_{\sigma-2}^m g_{\sigma-2}, t_{\sigma-2}^m \tilde{d}_p, t_{\sigma-2}^m \tilde{d}_{\sigma-1}, \quad m \in \mathbb{Z}^{\sigma+N-1}, g_{\sigma-2} \in \dot{\mathfrak{g}}_{\sigma-2}, p = \sigma, \dots, \sigma+N-1$span the subalgebra$\mathfrak{g}_{\sigma-1}$. The invariance of (4.18) under the action of $t_{\sigma-2}^m k_{\sigma-1}$ has already been confirmed.

We observe that the remaining generators of $\mathfrak{g}_{\sigma-1}$ annihilate 1 since they belong to $\mathfrak{g}_{\sigma-2}^{(+)}$. Furthermore, it is implied that the elements $t_{\sigma-2}^m k_p$ and $t_{\sigma-2}^m g_{\sigma-2}$ annihilate (5.18) because they commute with $t_{\sigma-2}^r k_{\sigma-1}$. (2.2) implies that $t_{\sigma-2}^m \tilde{d}_{\sigma-1}$ also annihilates (5.18).

Using the commutator relation $[t_{\sigma-2}^m \tilde{d}_p, t_{\sigma-2}^r k_{\sigma-1}] = r_p t_{\sigma-2}^{r+m} k_{\sigma-1}$, we get that

$$(t_{\sigma-2}^m \tilde{d}_p)\left((t_{\sigma-2}^{r_\sigma}k_{\sigma-1})\dots(t_{\sigma-2}^{r_s}k_{\sigma-1})\mathbf{1} - c^{s-1}(t_{\sigma-2}^{r_\sigma+\dots+r_s}k_{\sigma-1})\mathbf{1}\right)$$

$$= \sum_{j=\sigma}^{s} r_j\left((t_{\sigma-2}^{r_\sigma}k_{\sigma-1})\dots\left(t_{\sigma-2}^{r_j+m}k_{\sigma-1}\right)\dots(t_{\sigma-2}^{r_s}k_{\sigma-1})\mathbf{1} - c^{s-1}(t_{\sigma-2}^{r_1+\dots+r_s+m}k_{\sigma-1})\mathbf{1}\right)$$

And (5.18), on the right side. This demonstrates that $R(S)_{\sigma-1}$is provided by (5.18). This implies that the space of Laurent polynomials$T_{\sigma-1} = \mathbb{C}[q_\sigma^{\pm}, \dots, q_{\sigma+N-1}^{\pm}]$, under the isomorphism, may be identified with the top of the module $V_{\mathfrak{g}_{\sigma-2}}(c) = V_{\mathfrak{g}_{\sigma-2}}/R(S)$

$$(t_{\sigma-2}^r k_{\sigma-1})\mathbf{1} \mapsto c q_{\sigma-2}^r. \qquad (5.19)$$

Let us describe the action of the subalgebra $\mathfrak{g}_{\sigma-1}$ on the top $T_{\sigma-1}$. It follows from the relations (5.18) that

Next, we have seen that $t_{\sigma-2}^m k_p, t_{\sigma-2}^m g_{\sigma-2}$ and $t_{\sigma-2}^m \tilde{d}_{\sigma-1}$ annihilate $T_{\sigma-1}$:

$$(t_{\sigma-2}^m k_p)q_{\sigma-2}^r = 0, \quad (t_{\sigma-2}^m g_{\sigma-2})q_{\sigma-2}^r = 0, \quad (t_{\sigma-2}^m \tilde{d}_{\sigma-1})q_{\sigma-2}^r = 0 \qquad (5.20)$$

Since $t_{\sigma-2}^m \tilde{d}_{\sigma-1} = -t_{\sigma-2}^m d_{\sigma-1} + \frac{1}{2}(\mu+\nu)t_{\sigma-2}^m k_{\sigma-1}$, we get that

$$(t_{\sigma-2}^m d_{\sigma-1})q_{\sigma-2}^r = \frac{1}{2}(\mu+\nu)c q_{\sigma-2}^{r+m} \qquad (5.21)$$

Finally,

$$(t_{\sigma-2}^m \tilde{d}_p)q_{\sigma-2}^r = \frac{1}{c}(t_{\sigma-2}^m \tilde{d}_p)(t_{\sigma-2}^r k_{\sigma-1})\mathbf{1} = \frac{1}{c}[t_{\sigma-2}^m \tilde{d}_p, t_{\sigma-2}^r k_{\sigma-1}]\mathbf{1}$$

$$= \frac{1}{c}r_p(t_{\sigma-2}^{r+m}k_{\sigma-1})\mathbf{1} = r_p q_{\sigma-2}^{r+m} \qquad (5.22)$$

Taking into account that $t_{\sigma-2}^m \tilde{d}_p = t_{\sigma-2}^m d_p - m_p \nu t_{\sigma-2}^m k_{\sigma-1}$, we obtain

$$(t_{\sigma-2}^m d_p)q_{\sigma-2}^r = (r_p + \nu c m_p)q_{\sigma-2}^{r+m} (5.23)$$

This is equivalent to the tensor module action (3.6), with the trivial action of $sl_{\sigma+N-1}$and I functioning as multiplication by $(\sigma+N-1)\nu c$.. This completes the evidence for portion part (a). Part (b) follows from Theorem 4.3.3, and part (c) has been established previously. For part (d), we observe that $L(T_{\sigma-1})$ *is a unique irreducible factor of* $M(T_{\sigma-1})$; hence, the projection $M(T_{\sigma-1}) \to L(T_{\sigma-1})$ factors

through $V_{\mathfrak{g}_{\sigma-2}}(c)$ as in (5.17). Note that the kernel of the homomorphism $M(T_{\sigma-1}) \to V_{\mathfrak{g}_{\sigma-2}}(c)$ is the submodule given by $\{(t_{\sigma-1}^{-1}t_{\sigma-2}^{m}\tilde{d}_{\sigma-1})\mathbf{1} \mid m \in \mathbb{Z}^{\sigma+N-1}\}$.Using Theorem 4.3.3 again, we infer that $L(T_{\sigma-1})$ inherits the vertex operator algebra structure from $V_{\mathfrak{g}_{\sigma-2}}(c)$ as indicated by formula (4.11). This concludes the evidence of the Proposition.
Next, we investigate the structure of the VOA $L(T_{\sigma-1})$.

Theorem 5.3 (see [12]).

(a) The VOA $L(T_{\sigma-1})$ is generated by the following elements: $q_{\sigma-2}^{m} = \frac{1}{c}(t_{\sigma-2}^{m}k_{\sigma-1})\mathbf{1}, (t_{\sigma-1}^{-1}g_{\sigma-2})\mathbf{1}$, $(t_{\sigma-1}^{-1}k_a)\mathbf{1}, (t_{\sigma-1}^{-1}\tilde{d}_a)\mathbf{1}, E_{ab}, (t_{\sigma-1}^{-2}\tilde{d}_{\sigma-1})\mathbf{1}$, with $m \in \mathbb{Z}^{\sigma+N-1}, g_{\sigma-2} \in \dot{\mathfrak{g}}_{\sigma-2}, a, b = \sigma, \dots, \sigma+N-1$, where

$$E_{ab} = \frac{1}{c}(t_{\sigma-1}^{-1}t_{\sigma-2}^{\epsilon_a}\tilde{d}_b)(t_{\sigma-2}^{-\epsilon_a}k_{\sigma-1})\mathbf{1} - (t_{\sigma-1}^{-1}\tilde{d}_b)\mathbf{1} + \frac{1}{c}\delta_{ab}(t_{\sigma-1}^{-1}k_a)\mathbf{1} \qquad (5.24)$$

The a-th position of which has the value 1, and ϵ_a is a standard basis vector in $\mathbb{Z}^{\sigma+N-1}$.

(b) The algebra of Laurent polynomials can act on $T_{\sigma-1}$ via the $L(T_{\sigma-1})$ module, which has the structure of a $\mathbb{C}[q_\sigma^{\pm}, \dots, q_{\sigma+N-1}^{\pm}]$-module. The following vertex operator gives the operation of the field $k_{\sigma-1}(m,z)$:

$$\frac{1}{c}k_{\sigma-1}(m,z) = Y(q_{\sigma-2}^{m}, z) = q_{\sigma-2}^{m}\exp\left(\sum_{p=\sigma}^{\sigma+N-1} m_p \sum_{j\in\mathbb{Z}\setminus\{0\}} \frac{1}{cj}(t_{\sigma-1}^{-j}k_p)z^j\right). \qquad (5.25)$$

(c) The following is the expression for the action of the remaining fields in (5.1)–(5.3) on $L(T_{\sigma-1})$

$$g_{\sigma-2}(m,z) = g_{\sigma-2}(z)Y(q_{\sigma-2}^{m}, z), \qquad (5.26)$$

$$k_a(m,z) = k_a(z)Y(q_{\sigma-2}^{m}, z) \qquad (5.27)$$

$$\tilde{d}_a(m,z) =: \tilde{d}_a(z)Y(q_{\sigma-2}^{m}, z): + \sum_{p=\sigma}^{\sigma+N-1} m_p Y(E_{pa}, z)Y(q_{\sigma-2}^{m}, z), \qquad (5.28)$$

$$\tilde{d}_{\sigma-1}(m,z) =: \tilde{d}_{\sigma-1}(z)Y(q_{\sigma-2}^{m}, z): + \frac{1}{c}\sum_{a,b=\sigma}^{\sigma+N-1} m_a k_b(z)Y(E_{ab}, z)Y(q_{\sigma-2}^{m}, z)$$

$$+\left(\mu - \frac{1}{c}\right)\sum_{p=\sigma}^{\sigma+N-1} m_p\left(\frac{\partial}{\partial z}k_p(z)\right)Y(q_{\sigma-2}^{m}, z) \qquad (5.29)$$

(d) The tensor product of two VOAs is isomorphic to the vertex operator algebra $L(T_{\sigma-1})$:

$$L(T_{\sigma-1}) = V_{Hyp}^{+} \otimes L_{\mathfrak{f}}(\gamma_{\sigma-1}),$$

where $L_{\mathfrak{f}}(\gamma_{\sigma-1})$is the simple VOA corresponding to the twisted Virasoro-affine Lie algebra built from the reductive Lie algebra$\mathfrak{f} = \dot{\mathfrak{g}}_{\sigma-2} \oplus g_{\sigma-2}l_{\sigma+N-1}$, and V_{Hyp}^{+} is a sub-VOA of the lattice VOA discussed in Section 4.5. The central character $\gamma_{\sigma-1}$ is determined by the values listed below:

$$c_{\mathfrak{g}_{\sigma-2}} = c, \quad c_{sl_{\sigma+N-1}} = 1 - \mu c$$
$$c_{\mathcal{H}ei} = (\sigma + N - 1)(1 - \mu c) - (\sigma + N - 1)^2 \nu c, \quad c_{\mathcal{VH}} = (\sigma + N - 1)\left(\frac{1}{2} - \nu c\right),$$
$$c_{\mathcal{V}ir} = 12c(\mu + \nu) - 2(\sigma + N - 1). \qquad (5.30)$$

The fields $k_{\sigma-1}(m,z), k_p(z), \tilde{d}_p(z), p = \sigma, \dots, \sigma + N - 1$, act on $V_{Hyp}^{+}$by

$$k_{\sigma-1}(m,z) = cY(e^{mu}, z), \quad k_p(z) = cu_p(z), \quad \tilde{d}_p(z) = v_p(z)$$

Whereas $L_{\mathfrak{f}}(\gamma_{\sigma-1})$is affected by the fields$g_{\sigma-2}(z)$ and $Y(E_{ab}, z)$, the Virasoro field of the tensor product $V_{Hyp}^{+} \otimes L_{\mathfrak{f}}(\gamma_{\sigma-1})$is the field$\tilde{d}_{\sigma-1}(z)$.

Proof. The following approach will be used to ascertain the structure of the VOA $L(T_{\sigma-1})$. Any given homogeneous component of the kernel of the epimorphism $\pi: V_{\mathfrak{g}_{\sigma-2}}(c) \to L(T_{\sigma-1})$,can be calculated using the method developed in [3], though obviously this computation is only possible for components of low degree. A relation between the fields in $(T_{\sigma-1})$ may be obtained for every v$v \in \mathrm{Ker}\pi$:

$$Y_{L(T_{\sigma-1})}(v,z) = 0$$

It turns out that the structure of $L(T_{\sigma-1})$ as a VOA may be fully determined by knowing the elements of $\mathrm{Ker}(\pi)$ of degrees 1 and 2.

Let's use the following example to demonstrate the [3] approach. Fix $m \in \mathbb{Z}^{\sigma+N-1}$ and $g_{\sigma-2} \in \dot{\mathfrak{g}}_{\sigma-2}$. Consider the subspace $P_1 = \mathrm{Span}\langle (t_{\sigma-1}^{-1} t_{\sigma-2}^{r} g_{\sigma-2}) q_{\sigma-2}^{m-r} \mid r \in \mathbb{Z}^{\sigma+N-1} \rangle \subset V_{\mathfrak{g}_{\sigma-2}}(c)$. This subspace belongs to the homogeneous component of weight $(-1, m)$ in $V_{\mathfrak{g}_{\sigma-2}}(c)$. We are going to find the intersection of P_1 with Ker π. We note that a vector v in component $(-1, m)$ of $V_{\mathfrak{g}_{\sigma-2}}(c)$ belongs to Ker π if and only if $U_1((\mathfrak{g}_{\sigma-2})_+)v = 0$.

Lemma 5.4 (see [12]). Let $g_{\sigma-2}, g'_{\sigma-2} \in \dot{\mathfrak{g}}_{\sigma-2}, b = \sigma, \dots, \sigma + N - 1$. Then

$$(t_{\sigma-1} t_{\sigma-2}^{s} k_{\sigma-1})(t_{\sigma-1}^{-1} t_{\sigma-2}^{r} g_{\sigma-2}) q_{\sigma-2}^{m-r} = 0, \quad (t_{\sigma-1} t_{\sigma-2}^{s} k_b)(t_{\sigma-1}^{-1} t_{\sigma-2}^{r} g_{\sigma-2}) q_{\sigma-2}^{m-r} = 0, \qquad (5.31)$$

$$(t_{\sigma-1} t_{\sigma-2}^{s} g'_{\sigma-2})(t_{\sigma-1}^{-1} t_{\sigma-2}^{r} g_{\sigma-2}) q_{\sigma-2}^{m-r} = (g'_{\sigma-2} \mid g_{\sigma-2}) c q_{\sigma-2}^{m+s}, \qquad (5.32)$$

$$\left(t_{\sigma-1} t_{\sigma-2}^{s} \tilde{d}_{\sigma-1}\right)(t_{\sigma-1}^{-1} t_{\sigma-2}^{r} g_{\sigma-2}) q_{\sigma-2}^{m-r} = 0, \quad \left(t_{\sigma-1} t_{\sigma-2}^{s} \tilde{d}_b\right)(t_{\sigma-1}^{-1} t_{\sigma-2}^{r} g_{\sigma-2}) q_{\sigma-2}^{m-r} = 0. \qquad (5.33)$$

Proof. Let us prove (5.32):

$$(t_{\sigma-1} t_{\sigma-2}^{s} g'_{\sigma-2})(t_{\sigma-1}^{-1} t_{\sigma-2}^{r} g_{\sigma-2}) q_{\sigma-2}^{m-r} = [t_{\sigma-1} t_{\sigma-2}^{s} g'_{\sigma-2}, t_{\sigma-1}^{-1} t_{\sigma-2}^{r} g_{\sigma-2}] q_{\sigma-2}^{m-r} + (t_{\sigma-1}^{-1} t_{\sigma-2}^{r} g_{\sigma-2})(t_{\sigma-1} t_{\sigma-2}^{s} g'_{\sigma-2}) q_{\sigma-2}^{m-r}$$

Since $(\mathfrak{g}_{\sigma-2})_+$*acts trivially on top of* $V_{\mathfrak{g}_{\sigma-2}}(c)$, the second term disappears. We employ the relations in $\mathfrak{g}_{\sigma-2}$, (5.19), and (5.20) for the first term:

$$[t_{\sigma-1} t_{\sigma-2}^{s} g'_{\sigma-2}, t_{\sigma-1}^{-1} t_{\sigma-2}^{r} g_{\sigma-2}] q_{\sigma-2}^{m-r}$$
$$= (t_{\sigma-2}^{r+s}[g'_{\sigma-2}, g_{\sigma-2}]) q_{\sigma-2}^{m-r} + (g'_{\sigma-2} \mid g_{\sigma-2})(t_{\sigma-2}^{r+s} k_{\sigma-1}) q_{\sigma-2}^{m-r} + (g'_{\sigma-2} \mid g_{\sigma-2}) \sum_{p=\sigma}^{\sigma+N-1} s_p \left(t_{\sigma-2}^{r+s} k_p\right) q_{\sigma-2}^{m-r}.$$
$$= (g'_{\sigma-2} \mid g_{\sigma-2}) c q_{\sigma-2}^{m+s}$$

The same method is used to obtain the other equalities in this Lemma's assertion. Let's examine this Lemma's outcomes. The right-hand sides in (5.31) and (5.33) are shown to be independent of r . Thus $U_1((\mathfrak{g}_{\sigma-2})_+)\big((t_{\sigma-1}^{-1}t_{\sigma-2}^{r}g_{\sigma-2})q_{\sigma-2}^{m-r} - (t_{\sigma-1}^{-1}g_{\sigma-2})q_{\sigma-2}^{m}\big) = 0$, which impLies that $(t_{\sigma-1}^{-1}t_{\sigma-2}^{r}g_{\sigma-2})q_{\sigma-2}^{m-r} - (t_{\sigma-1}^{-1}g_{\sigma-2})q_{\sigma-2}^{m} \in \mathrm{Ker}\pi$.We find that the following relation exists in $(T_{\sigma-1})$ by using the state–field correspondence:

$$:Y(t_{\sigma-1}^{-1}t_{\sigma-2}^{r}g_{\sigma-2},z)Y(q_{\sigma-2}^{m-r},z):=:Y(t_{\sigma-1}^{-1}g_{\sigma-2},z)Y(q_{\sigma-2}^{m},z): \quad (5.34)$$

We can omit the standard ordering symbol because these vertex operators commute. Specifically, for $m=r$, we obtain that:

$$Y(t_{\sigma-1}^{-1}t_{\sigma-2}^{m}g_{\sigma-2},z) = Y(t_{\sigma-1}^{-1}g_{\sigma-2},z)Y(q_{\sigma-2}^{m},z) \quad (5.35)$$

The fields $Y(t_{\sigma-1}^{-1}t_{\sigma-2}^{m}g_{\sigma-2},z) = g_{\sigma-2}(m,z)$ and $Y(q_{\sigma-2}^{m},z)$ reduce to the more basic fields $Y(t_{\sigma-1}^{-1}g_{\sigma-2},z)$ due to this factorization property; the affine subalgebra $\mathbb{C}[t_{\sigma-1},t_{\sigma-1}^{-1}]\otimes\dot{\mathfrak{g}}_{\sigma-2}\oplus\mathbb{C}k_{\sigma-1}\subset\mathfrak{g}_{\sigma-2}$ is represented by the fields $g_{\sigma-2}(z)$, and we provide comparable factorization formulas for additional fields below. Additionally, with the exception of $Y\big(t_{\sigma-1}^{-2}\tilde{d}_{\sigma-1},z\big)$, *the* affine fields $Y(t_{\sigma-1}^{-1}g_{\sigma-2},z) = g_{\sigma-2}(z)$ commute with other elementary fields, which suggests that when c is not the critical level for this affine subalgebra, the affine VOA produced by the affine fields splits off as a tensor factor in $L(T_{\sigma-1})$, and the tensor product decomposition of $L(T_{\sigma-1})$ will be obtained in this manner.

The vertex operators $Y(t_{\sigma-1}^{-1}t_{\sigma-2}^{r}k_a,z)$ and $Y(t_{\sigma-2}^{r}k_{\sigma-1},z)$ can now be obtained using the formula. For a fixed $m\in\mathbb{Z}^{\sigma+N-1}$ and $\sigma\le a\le\sigma+N-1$, consider the subspace $P_2=\mathrm{Span}\langle(t_{\sigma-1}^{-1}t_{\sigma-2}^{r}k_a)q_{\sigma-2}^{m-r}\mid r\in\mathbb{Z}^{\sigma+N-1}\rangle\subset V_{\mathfrak{g}_{\sigma-2}}(c)$. Once more, we will determine where P_2 and $\mathrm{Ker}\pi$ intersect.

Lemma 5.5 [12]. Let $\sigma\le a,b\le\sigma+N-1$. Then

$$(t_{\sigma-1}t_{\sigma-2}^{s}k_{\sigma-1})(t_{\sigma-1}^{-1}t_{\sigma-2}^{r}k_a)q_{\sigma-2}^{m-r}=0,\quad (t_{\sigma-1}t_{\sigma-2}^{s}k_b)(t_{\sigma-1}^{-1}t_{\sigma-2}^{r}k_a)q_{\sigma-2}^{m-r} = 0, \quad (5.36)$$

$$(t_{\sigma-1}t_{\sigma-2}^{s}g_{\sigma-2})(t_{\sigma-1}^{-1}t_{\sigma-2}^{r}k_a)q_{\sigma-2}^{m-r}=0,\quad g_{\sigma-2}\in\dot{\mathfrak{g}}_{\sigma-2},$$

$$\big(t_{\sigma-1}t_{\sigma-2}^{s}\tilde{d}_{\sigma-1}\big)(t_{\sigma-1}^{-1}t_{\sigma-2}^{r}k_a)q_{\sigma-2}^{m-r}=0, \quad (5.37)$$

$$\big(t_{\sigma-1}t_{\sigma-2}^{s}\tilde{d}_b\big)(t_{\sigma-1}^{-1}t_{\sigma-2}^{r}k_a)q_{\sigma-2}^{m-r}=c\delta_{ab}q_{\sigma-2}^{m+s} \quad (5.38)$$

We skip these computations and use the same evidence for this Lemma as for Lemma 5.4.

The right-hand sides in (5.36)–(5.38) are once more shown to be independent of r.Thus $(t_{\sigma-1}^{-1}t_{\sigma-2}^{r}k_a)q_{\sigma-2}^{m-r}-(t_{\sigma-1}^{-1}k_a)q_{\sigma-2}^{m}\in\mathrm{Ker}\pi$, or equivalently,

$$(t_{\sigma-1}^{-1}t_{\sigma-2}^{r}k_a)q_{\sigma-2}^{m-r}=(t_{\sigma-1}^{-1}k_a)q_{\sigma-2}^{m}\quad \text{in}\quad L(T_{\sigma-1}) \quad (5.39)$$

The following relation obtains in $(T_{\sigma-1})$ when $r=m$ and the state–field correspondence is applied:

$$Y(t_{\sigma-1}^{-1}t_{\sigma-2}^{m}k_a,z)=Y(t_{\sigma-1}^{-1}k_a,z)Y(q_{\sigma-2}^{m},z)(5.40)$$

We removed the standard ordering symbol on the right side because these vertex operators commute.Also, taking into account that $t_{\sigma-1}^{-1}t_{\sigma-2}^{m}k_{\sigma-1}=\sum_{p=\sigma}^{\sigma+N-1}m_p t_{\sigma-1}^{-1}t_{\sigma-2}^{m}k_p$, we obtain

$$c\frac{\partial}{\partial z}Y(q_{\sigma-2}^{m},z)=Y(t_{\sigma-1}^{-1}t_{\sigma-2}^{m}k_{\sigma-1},z)=\sum_{p=\sigma}^{\sigma+N-1}m_pY\big(t_{\sigma-1}^{-1}k_p,z\big)Y(q_{\sigma-2}^{m},z)(5.41)$$

The outcomes of Section 3 of [3] will thereafter be used. The vertex operator $Y(q^m_{\sigma-2}, z)$ is given by the formula (5.25), and it is demonstrated there that $L(T_{\sigma-1})$ is a module over a commutative associative algebra $\mathbb{C}[q^{\pm}_{\sigma}, \dots, q^{\pm}_{\sigma+N-1}]$. The compatibility of this formula with (5.41), as well as the ease with which (5.25) may be derived from (5.41), are evident.

In $L(T_{\sigma-1})$, we will eventually require another relation that can be obtained from (5.39) or (5.25):

$$(t^{-1}_{\sigma-1}t^{r}_{\sigma-2}k_{\sigma-1})q^{s}_{\sigma-2} = \sum_{p=\sigma}^{\sigma+N-1} r_p\big(t^{-1}_{\sigma-1}k_p\big)q^{r+s}_{\sigma-2} \tag{5.42}$$

For the vertex operator $Y\big(t^{-1}_{\sigma-1}t^{m}_{\sigma-2}\tilde{d}_a, z\big)$, we obtain a formula. We'll employ the same tactic as before.

Lemma 5.6 (see [12]). Let $\sigma \le a, b \le \sigma + N - 1$. Then

$$(t_{\sigma-1}t^{s}_{\sigma-2}k_{\sigma-1})\big(t^{-1}_{\sigma-1}t^{r}_{\sigma-2}\tilde{d}_a\big)q^{m-r}_{\sigma-2} = -s_a c q^{m+s}_{\sigma-2}, \tag{5.43}$$

$$\big(t_{\sigma-1}t^{s}_{\sigma-2}k_b\big)\big(t^{-1}_{\sigma-1}t^{r}_{\sigma-2}\tilde{d}_a\big)q^{m-r}_{\sigma-2} = \delta_{ab} c q^{m+s}_{\sigma-2}, \tag{5.44}$$

$$(t_{\sigma-1}t^{s}_{\sigma-2}g_{\sigma-2})\big(t^{-1}_{\sigma-1}t^{r}_{\sigma-2}\tilde{d}_a\big)q^{m-r}_{\sigma-2} = 0, \quad g_{\sigma-2} \in \dot{\mathfrak{g}}_{\sigma-2}, \tag{5.45}$$

$$\big(t_{\sigma-1}t^{s}_{\sigma-2}\tilde{d}_{\sigma-1}\big)\big(t^{-1}_{\sigma-1}t^{r}_{\sigma-2}\tilde{d}_a\big)q^{m-r}_{\sigma-2} = \big((m_a - r_a) - 2(\mu s_a - \nu r_a)c\big)q^{m+s}_{\sigma-2}, \tag{5.46}$$

$$\begin{aligned}&\big(t_{\sigma-1}t^{s}_{\sigma-2}\tilde{d}_b\big)\big(t^{-1}_{\sigma-1}t^{r}_{\sigma-2}\tilde{d}_a\big)q^{m-r}_{\sigma-2}\\ &\quad = \big(r_b(m_a - r_a) - s_a(m_b - r_b) - (\mu r_b s_a + \nu r_a s_b)c\big)q^{m+s}_{\sigma-2}.\end{aligned} \tag{5.47}$$

Proof. Let's demonstrate the math for (4.46) and leave the remainder to the reader as an exercise. Equations (5.7), (5.19), and (5.22) will be used:

$$\begin{aligned}\big(t_{\sigma-1}t^{s}_{\sigma-2}\tilde{d}_{\sigma-1}\big)\big(t^{-1}_{\sigma-1}t^{r}_{\sigma-2}\tilde{d}_a\big)q^{m-r}_{\sigma-2} &= \big[t_{\sigma-1}t^{s}_{\sigma-2}\tilde{d}_{\sigma-1}, t^{-1}_{\sigma-1}t^{r}_{\sigma-2}\tilde{d}_a\big]q^{m-r}_{\sigma-2}\\ &= \big(t^{r+s}_{\sigma-2}\tilde{d}_a\big)q^{m-r}_{\sigma-2} - s_a\big(t^{r+s}_{\sigma-2}\tilde{d}_{\sigma-1}\big)q^{m-r}_{\sigma-2}\\ +(-2\mu s_a + 2\nu r_a)(t^{r+s}_{\sigma-2}k_{\sigma-1})q^{m-r}_{\sigma-2} &- (-\mu s_a + 2\nu r_a)\sum_{p=\sigma}^{\sigma+N-1} r_p\big(t^{r+s}_{\sigma-2}k_p\big)q^{m-r}_{\sigma-2}\\ &= (m_a - r_a - 2\mu c s_a + 2\nu c r_a)q^{m+s}_{\sigma-2}\end{aligned}$$

The right-hand sides in Lemma 5.6 do depend on r, in contrast to the earlier examples. We can distinguish between constant, linear, and quadratic components, but keep in mind that this dependency is polynomial. Setting $r = 0$ obviously yields the constant term, which is generated by the element $\big(t^{-1}_{\sigma-1}\tilde{d}_a\big)q^{m}_{\sigma-2}$.

Upon comparing (5.43)–(5.47) with (5.36)–(5.38), we observe that the vector $-\frac{r_a}{c}\sum_{p=\sigma}^{\sigma+N-1} r_p\big(t^{-1}_{\sigma-1}k_p\big)q^{m}_{\sigma-2}$ provides the quadratic term :

$$\big(t_{\sigma-1}t^{s}_{\sigma-2}\tilde{d}_b\big)\left(-\frac{r_a}{c}\sum_{p=\sigma}^{\sigma+N-1} r_p\big(t^{-1}_{\sigma-1}k_p\big)q^{m}_{\sigma-2}\right) = -r_a r_b q^{m+s}_{\sigma-2}$$

While this vector is destroyed by the other raising operators.

Lastly, the vector $\sum_{p=\sigma}^{\sigma+N-1} r_p E^{m}_{pa}$ provides the linear-in-r component, where

$$E^{m}_{pa} = \big(t^{-1}_{\sigma-1}t^{\epsilon_p}_{\sigma-2}\tilde{d}_a\big)q^{m-\epsilon_p}_{\sigma-2} - \big(t^{-1}_{\sigma-1}\tilde{d}_a\big)q^{m}_{\sigma-2} + \frac{1}{c}\delta_{ap}\big(t^{-1}_{\sigma-1}k_p\big)q^{m}_{\sigma-2}. \tag{5.48}$$

Lemmas 5.6 and 5.5 make it clear to us that:

$$(t_{\sigma-1}t_{\sigma-2}^{s}k_{\sigma-1})\sum_{p=\sigma}^{\sigma+N-1} r_pE_{pa}^{m}=0,\quad (t_{\sigma-1}t_{\sigma-2}^{s}k_{b})\sum_{p=\sigma}^{\sigma+N-1} r_pE_{pa}^{m}=0,$$

$$(t_{\sigma-1}t_{\sigma-2}^{s}g_{\sigma-2})\sum_{p=\sigma}^{\sigma+N-1} r_pE_{pa}^{m}=0, \qquad (5.49)$$

$$\left(t_{\sigma-1}t_{\sigma-2}^{s}\tilde{d}_{\sigma-1}\right)\sum_{p=\sigma}^{\sigma+N-1} r_pE_{pa}^{m}=r_a(-1+2\nu c)q_{\sigma-2}^{m+s} \qquad (5.50)$$

$$\left(t_{\sigma-1}t_{\sigma-2}^{s}\tilde{d}_{b}\right)\sum_{p=\sigma}^{\sigma+N-1} r_pE_{pa}^{m}=(r_bm_a+(1-\mu c)s_ar_b-\nu cr_as_b)q_{\sigma-2}^{m+s}. \qquad (5.51)$$

The calculations above indicate that $U_1((\mathfrak{g}_{\sigma-2})_+)$ annihilates the following vector, which vanishes in $(T_{\sigma-1})$:

$$(t_{\sigma-1}^{-1}t_{\sigma-2}^{r}\tilde{d}_a)q_{\sigma-2}^{m-r}-(t_{\sigma-1}^{-1}\tilde{d}_a)q_{\sigma-2}^{m}-\sum_{p=\sigma}^{\sigma+N-1} r_pE_{pa}^{m}+\frac{r_a}{c}\sum_{p=\sigma}^{\sigma+N-1} r_p(t_{\sigma-1}^{-1}k_p)q_{\sigma-2}^{m}$$

$$=0. \qquad (5.52)$$

Applying the map Y and setting $r=m$ yields the following relation for the vertex operators in $L(T_{\sigma-1})$:

$$\tilde{d}_a(m,z)=:\tilde{d}_a(z)Y(q_{\sigma-2}^{m},z):+\sum_{p=\sigma}^{\sigma+N-1} m_pY\left(E_{pa}^{m},z\right)-\frac{m_a}{c}\sum_{p=\sigma}^{\sigma+N-1} m_pk_p(z)Y(q_{\sigma-2}^{m},z) \qquad (5.53)$$

Now let's create a relationship between E_{ab}^{m} and:

$$E_{ab}=E_{ab}^{0}=\left(t_{\sigma-1}^{-1}t_{\sigma-2}^{\epsilon_a}\tilde{d}_b\right)q_{\sigma-2}^{-\epsilon_a}-\left(t_{\sigma-1}^{-1}\tilde{d}_b\right)\mathbf{1}+\frac{1}{c}\delta_{ab}(t_{\sigma-1}^{-1}k_a)\mathbf{1} \qquad (5.54)$$

Lemma 5.7 (see [12]). (a) The fields $Y(q_{\sigma-2}^{m},z), g_{\sigma-2}(z), k_p(z), \tilde{d}_p(z)$ commute with $(E_{ab},z), p,a,b=\sigma,\dots,\sigma+N-1$.

(b) The relationship that follows is valid:

$$E_{ab}^{m}=(E_{ab})_{(-1)}q_{\sigma-2}^{m}+\frac{m_b}{c}(t_{\sigma-1}^{-1}k_a)q_{\sigma-2}^{m}. \qquad (5.55)$$

Proof. Let's demonstrate that

$$(q_{\sigma-2}^{m})_{(n)}E_{ab}=0 \text{ for all } n\geq 0 \qquad (5.56)$$

Since $\deg\left((q_{\sigma-2}^{m})_{(n)}E_{ab}\right)=-n$, Only the case when $n=0$ needs to be taken into account. But $(q_{\sigma-2}^{m})_{(0)}=\frac{1}{c}(t_{\sigma-1}t_{\sigma-2}^{m}k_{\sigma-1})$, and we obtain the intended assertion from (5.49). We can now determine that the fields $Y(q_{\sigma-2}^{m},z)$ and $Y(E_{ab},z)$ commute by using the commutator formula (4.2). We may infer from (5.49)-(5.51) that $g_{\sigma-2}(z), k_p(z)$ and $\tilde{d}_p(z)$ likewise commute with $Y(E_{ab},z)$ by using a similar approach.

Considering the skew symmetry identity (4.7), we derive the equality as a result of (5.56).

$$(E_{ab})_{(-1)}q_{\sigma-2}^{m}=(q_{\sigma-2}^{m})_{(-1)}E_{ab}$$

When we replace (4.54) on the right-hand side of this equality, we obtain:

$$(E_{ab})_{(-1)}q_{\sigma-2}^{m}=\frac{1}{c}(t_{\sigma-2}^{m}k_{\sigma-1})\left((t_{\sigma-1}^{-1}t_{\sigma-2}^{\epsilon_a}\tilde{d}_b)q_{\sigma-2}^{-\epsilon_a}-(t_{\sigma-1}^{-1}\tilde{d}_b)\mathbf{1}+\frac{1}{c}\delta_{ab}(t_{\sigma-1}^{-1}k_a)\mathbf{1}\right)$$

$$=\frac{1}{c}(t_{\sigma-1}^{-1}t_{\sigma-2}^{\epsilon_a}\tilde{d}_b)(t_{\sigma-2}^{m}k_{\sigma-1})q_{\sigma-2}^{-\epsilon_a}-\frac{1}{c}(t_{\sigma-1}^{-1}\tilde{d}_b)(t_{\sigma-2}^{m}k_{\sigma-1})\mathbf{1}+\frac{1}{c^2}\delta_{ab}(t_{\sigma-1}^{-1}k_a)(t_{\sigma-2}^{m}k_{\sigma-1})\mathbf{1}$$

$$-\frac{1}{c}[t_{\sigma-1}^{-1}t_{\sigma-2}^{\epsilon_a}\tilde{d}_b,t_{\sigma-2}^{m}k_{\sigma-1}]q_{\sigma-2}^{-\epsilon_a}+\frac{1}{c}[t_{\sigma-1}^{-1}\tilde{d}_b,t_{\sigma-2}^{m}k_{\sigma-1}]\mathbf{1}$$

$$=(t_{\sigma-1}^{-1}t_{\sigma-2}^{\epsilon_a}\tilde{d}_b)q_{\sigma-2}^{m-\epsilon_a}-(t_{\sigma-1}^{-1}\tilde{d}_b)q_{\sigma-2}^{m}+\frac{1}{c}\delta_{ab}(t_{\sigma-1}^{-1}k_a)q_{\sigma-2}^{m}-\frac{m_b}{c}(t_{\sigma-1}^{-1}t_{\sigma-2}^{m+\epsilon_a}k_{\sigma-1})q_{\sigma-2}^{-\epsilon_a}+\frac{m_b}{c}(t_{\sigma-1}^{-1}t_{\sigma-2}^{m}k$$

$$=E_{ab}^{m}-\frac{m_b}{c}(t_{\sigma-1}^{-1}k_a)q_{\sigma-2}^{m}.$$

In order to obtain the final equality, we utilized (5.48) and (5.42). This concludes the lemma's proof.

When (5.53) and (5.55) are combined, we get (5.28). Additionally, we learn from (5.52) and (5.55) that.

$$(t_{\sigma-1}^{-1}t_{\sigma-2}^{r}\tilde{d}_a)q_{\sigma-2}^{m}$$
$$=(t_{\sigma-1}^{-1}\tilde{d}_a)q_{\sigma-2}^{m+r}+\sum_{p=\sigma}^{\sigma+N-1}r_p(E_{pa})_{(-1)}q_{\sigma-2}^{m+r}$$
$$+\frac{m_a}{c}\sum_{p=\sigma}^{\sigma+N-1}r_p(t_{\sigma-1}^{-1}k_p)q_{\sigma-2}^{m+r}\quad(5.57)$$

The commutator relations between $Y(E_{ab},z),\sigma\le a,b\le\sigma+N-1$ are then ascertained.

Lemma 5.8 (see [12]). (a) $(E_{ab})_{(0)}E_{sp}=\delta_{bs}E_{ap}-\delta_{ap}E_{sb}$,

(b) $(E_{ab})_{(1)}E_{sp}=(1-\mu c)\delta_{bs}\delta_{ap}\mathbf{1}-\nu c\delta_{ab}\delta_{sp}\mathbf{1}$,

(c) $(E_{ab})_{(n)}E_{sp}=0$ for $n\ge 2$.

Proof. Let us perform the calculations for part (b) of the Lemma. Since $\deg(E_{ab})=1$, we get that $\deg\big((E_{ab})_{(n)}E_{sp}\big)=1-n$, from which part (c) easily follows.

$$(E_{ab})_{(1)}E_{sp}=\left((t_{\sigma-1}^{-1}t_{\sigma-2}^{\epsilon_a}\tilde{d}_b)_{(-1)}q_{\sigma-2}^{-\epsilon_a}\right)_{(1)}E_{sp}-(t_{\sigma-1}\tilde{d}_b)E_{sp}+\frac{1}{c}\delta_{ab}(t_{\sigma-1}k_a)E_{sp}(5.58)$$

The final pair of terms disappear by (5.51) and (5.49). We apply the Borcherds' identity (4.6) to evaluate the first term on the right side of (5.58). Observing that for any $n\ge 0$, by (5.56), $\left(q_{\sigma-2}^{-\epsilon_a}\right)_{(n)}E_{sp}=0$, we obtain:

$$\left((t_{\sigma-1}^{-1}t_{\sigma-2}^{\epsilon_a}\tilde{d}_b)_{(-1)}q_{\sigma-2}^{-\epsilon_a}\right)_{(1)}E_{sp}$$
$$=\frac{1}{c}(t_{\sigma-1}t_{\sigma-2}^{-\epsilon_a}k_{\sigma-1})(t_{\sigma-2}^{\epsilon_a}\tilde{d}_b)E_{sp}+\frac{1}{c}(t_{\sigma-2}^{-\epsilon_a}k_{\sigma-1})(t_{\sigma-1}t_{\sigma-2}^{\epsilon_a}\tilde{d}_b)E_{sp}$$

Since the first term equals zero

$$\frac{1}{c}(t_{\sigma-1}t_{\sigma-2}^{-\epsilon_a}k_{\sigma-1})(t_{\sigma-2}^{\epsilon_a}\tilde{d}_b)E_{sp}$$
$$=\frac{1}{c}(t_{\sigma-2}^{\epsilon_a}\tilde{d}_b)(t_{\sigma-1}t_{\sigma-2}^{-\epsilon_a}k_{\sigma-1})E_{sp}-\frac{1}{c}[t_{\sigma-2}^{\epsilon_a}\tilde{d}_b,t_{\sigma-1}t_{\sigma-2}^{-\epsilon_a}k_{\sigma-1}]E_{sp}$$

Then we may apply (4.49) to the right-hand side. Lastly, by using (4.51) and (4.19), we obtain:

$$\frac{1}{c}\left(t_{\sigma-2}^{-\epsilon_a}k_{\sigma-1}\right)\left(t_{\sigma-1}t_{\sigma-2}^{\epsilon_a}\tilde{d}_b\right)E_{sp} = \left((1-\mu c)\delta_{bs}\delta_{ap} - \nu c\delta_{ab}\delta_{sp}\right)\mathbf{1}$$

We leave portion (a) of the Lemma as an exercise for the reader, and thus conclude the proof of part (b).

We determine that the operators $(E_{ab})_{(n)}$yield a representation of affine$\widehat{g_{\sigma-2}l}_{\sigma+N-1}$ by comparing Lemmas 5.8 and 4.4.2(a).

Lemma 5.9 (see [12]). The following relations hold in $(T_{\sigma-1})$:

$(a)\left(t_{\sigma-1}^{-1}t_{\sigma-2}^{r}\tilde{d}_{\sigma-1}\right)q_{\sigma-2}^{m} = \frac{1}{c}\sum_{p=\sigma}^{\sigma+N-1} m_p\left(t_{\sigma-1}^{-1}k_p\right)q_{\sigma-2}^{m+r},$

$(b)\quad \left(t_{\sigma-1}t_{\sigma-2}^{r}\tilde{d}_{\sigma-1}\right)(E_{ab})_{(-1)}q_{\sigma-2}^{m} = \delta_{ab}(-1+2\nu c).$

Proof.

$$\left(t_{\sigma-1}^{-1}t_{\sigma-2}^{r}\tilde{d}_{\sigma-1}\right)q_{\sigma-2}^{m} = \frac{1}{c}\left(t_{\sigma-1}^{-1}t_{\sigma-2}^{r}\tilde{d}_{\sigma-1}\right)\left(t_{\sigma-2}^{m}k_{\sigma-1}\right)\mathbf{1} = \frac{1}{c}\left[t_{\sigma-1}^{-1}t_{\sigma-2}^{r}\tilde{d}_{\sigma-1}, t_{\sigma-2}^{m}k_{\sigma-1}\right]\mathbf{1}$$
$$= \frac{1}{c}\sum_{p=\sigma}^{\sigma+N-1} m_p\left(t_{\sigma-1}^{-1}t_{\sigma-2}^{r+m}k_p\right)\mathbf{1} = \frac{1}{c}\sum_{p=\sigma}^{\sigma+N-1} m_p\left(t_{\sigma-1}^{-1}k_p\right)q_{\sigma-2}^{r+m}.$$

This validates the assertion (a). From Lemma 5.7(b), (5.50), and (5.37), part (b) is inferred.

Let us now examine the characteristics of the field $\tilde{d}_{\sigma-1}(r,z)$. This will necessitate computations using certain degree 2 components. Let$\deg(v) = 2$ *and* $v \in L(T_{\sigma-1})$. Any non-zero vector can generate $L(T_{\sigma-1})$, as it is an irreducible $\mathfrak{g}_{\sigma-2}$-module. $U_2((\mathfrak{g}_{\sigma-2})_+)v = T_{\sigma-1}$if If $v \neq 0$. Nonetheless, it is evident that $\mathfrak{g}_\sigma$ is the source of $(\mathfrak{g}_{\sigma-2})_+$. This implies that *v = 0* since$\mathfrak{g}_\sigma v = 0$. An equation for the field $\tilde{d}_{\sigma-1}(m,z)$will be obtained by using this observation to identify a relation in$L(T_{\sigma-1})$ involving $\left(t_{\sigma-1}^{-2}t_{\sigma-2}^{m}\tilde{d}_{\sigma-1}\right)\mathbf{1}$.

Lemma 5.10 (see [12]). The following relation holds in $(T_{\sigma-1})$:

$$\left(t_{\sigma-1}^{-2}t_{\sigma-2}^{m}\tilde{d}_{\sigma-1}\right)\mathbf{1} = \left(t_{\sigma-1}^{-2}\tilde{d}_{\sigma-1}\right)q_{\sigma-2}^{m} + \frac{1}{c}\sum_{p,j=\sigma}^{\sigma+N-1} m_p\left(t_{\sigma-1}^{-1}k_j\right)\left(E_{pj}\right)_{(-1)}q_{\sigma-2}^{m}$$
$$-\frac{1}{c}(1-\mu c)\sum_{p=\sigma}^{\sigma+N-1} m_p\left(t_{\sigma-1}^{-2}k_p\right)q_{\sigma-2}^{m}$$

Proof. This Lemma will be demonstrated by demonstrating that the vector.

$$v = \left(t_{\sigma-1}^{-2}t_{\sigma-2}^{m}\tilde{d}_{\sigma-1}\right)\mathbf{1} - \left(t_{\sigma-1}^{-2}\tilde{d}_{\sigma-1}\right)q_{\sigma-2}^{m} - \frac{1}{c}\sum_{p,j=\sigma}^{\sigma+N-1} m_p\left(t_{\sigma-1}^{-1}k_j\right)\left(E_{pj}\right)_{(-1)}q_{\sigma-2}^{m} + \frac{1}{c}(1$$
$$-\mu c)\sum_{p=\sigma}^{\sigma+N-1} m_p\left(t_{\sigma-1}^{-2}k_p\right)q_{\sigma-2}^{m}$$

is annihilated in $L(T_{\sigma-1})$ by $\mathfrak{g}_\sigma$. Let us show that $(t_{\sigma-1}t^s_{\sigma-2}\tilde{d}_{\sigma-1})v = 0$ in $(T_{\sigma-1})$:

$$(t_{\sigma-1}t^s_{\sigma-2}\tilde{d}_{\sigma-1})v = [t_{\sigma-1}t^s_{\sigma-2}\tilde{d}_{\sigma-1}, t^{-2}_{\sigma-1}t^m_{\sigma-2}\tilde{d}_{\sigma-1}]\mathbf{1} - [t_{\sigma-1}t^s_{\sigma-2}\tilde{d}_{\sigma-1}, t^{-2}_{\sigma-1}\tilde{d}_{\sigma-1}]q^m_{\sigma-2}$$

$$-\frac{1}{c}\sum_{p,j=\sigma}^{\sigma+N-1} m_p[t_{\sigma-1}t^s_{\sigma-2}\tilde{d}_{\sigma-1}, t^{-1}_{\sigma-1}k_j](E_{pj})_{(-1)}q^m_{\sigma-2} - \frac{1}{c}\sum_{p,j=\sigma}^{\sigma+N-1} m_p(t^{-1}_{\sigma-1}k_j)(t_{\sigma-1}t^s_{\sigma-2}\tilde{d}_{\sigma-1})(E_{pj})_{(-1)}q^m_{\sigma-2}$$

$$+\frac{1}{c}(1-\mu c)\sum_{p=\sigma}^{\sigma+N-1} m_p[t_{\sigma-1}t^s_{\sigma-2}\tilde{d}_{\sigma-1}, t^{-2}_{\sigma-1}k_p]q^m_{\sigma-2}$$

$$= 3(t^{-1}_{\sigma-1}t^{m+s}_{\sigma-2}\tilde{d}_{\sigma-1})\mathbf{1} + 4(\mu+\nu)(t^{-1}_{\sigma-1}t^{m+s}_{\sigma-2}k_{\sigma-1})\mathbf{1} - 2(\mu+\nu)\sum_{p=\sigma}^{\sigma+N-1} m_p(t^{-1}_{\sigma-1}t^{m+s}_{\sigma-2}k_p)\mathbf{1}$$

$$-3(t^{-1}_{\sigma-1}t^s_{\sigma-2}\tilde{d}_{\sigma-1})q^m_{\sigma-2} - 4(\mu+\nu)(t^{-1}_{\sigma-1}t^s_{\sigma-2}k_{\sigma-1})q^m_{\sigma-2}$$

$$-\frac{1}{c}\sum_{p,j=\sigma}^{\sigma+N-1} m_p(t^s_{\sigma-2}k_j)(E_{pj})(-1)q^m_{\sigma-2} - \frac{1}{c}(-1+2\nu c)\sum_{p=\sigma}^{\sigma+N-1} m_p(t^{-1}_{\sigma-1}k_p)q^{m+s}_{\sigma-2}$$

$$+\frac{2}{c}(1-\mu c)\sum_{p=\sigma}^{\sigma+N-1} m_p(t^{-1}_{\sigma-1}t^s_{\sigma-2}k_p)q^m_{\sigma-2}$$

$$= 4(\mu+\nu)\sum_{p=\sigma}^{\sigma+N-1}(m_p+s_p)(t^{-1}_{\sigma-1}k_p)q^{m+s}_{\sigma-2} - 2(\mu+\nu)\sum_{p=\sigma}^{\sigma+N-1} m_p(t^{-1}_{\sigma-1}k_p)q^{m+s}_{\sigma-2} - \frac{3}{c}\sum_{p=\sigma}^{\sigma+N-1} m_p(t^{-1}_{\sigma-1}k_p)q^{m}_{\sigma}$$

$$-4(\mu+\nu)\sum_{p=\sigma}^{\sigma+N-1} s_p(t^{-1}_{\sigma-1}k_p)q^{m+s}_{\sigma-2} - \frac{1}{c}\sum_{p,j=\sigma}^{\sigma+N-1} m_p(E_{pj})_{(-1)}(t^s_{\sigma-2}k_j)q^m_{\sigma-2}$$

$$-\frac{1}{c}(-1+2\nu c)\sum_{p,j=\sigma}^{\sigma+N-1} m_p(t^{-1}_{\sigma-1}k_p)q^{m+s}_{\sigma-2} + \frac{2}{c}(1-\mu c)\sum_{p=\sigma}^{\sigma+N-1} m_p(t^{-1}_{\sigma-1}k_p)q^{m+s}_{\sigma-2} = 0.$$

Similar treatment is given to the situations of additional items spanning $\mathfrak{g}_\sigma$. The formula (5.29) may be obtained by applying the state-field correspondence Y to both sides of the equality in Lemma 5.10. Let's finish Theorem 5.3's proof. All of the relationships between the fields mentioned in the Theorem's section (c) have now been established. The fields (5.1)-(5.3) create the universal enveloping vertex algebra $V_{\mathfrak{g}_{\sigma-2}}$.. Since $L(T_{\sigma-1})$ is a factor vertex algebra of $V_{\mathfrak{g}_{\sigma-2}}$, the same is true for $L(T_{\sigma-1})$. The assertion of part (a) of the Theorem is derived by considering the relations of part (c). As previously stated, the results of [3] support the assertion of section (b).

Let's demonstrate the claim stated in the theorem's component (d). First, a homomorphism between vertex algebras is constructed.

$$\varphi: \quad V^+_{Hyp} \to L(T_{\sigma-1}),$$

defined by $\varphi(e^{mu}) = q^m_{\sigma-2}, \varphi(u_p(-1)\mathbf{1}) = \frac{1}{c}(t^{-1}_{\sigma-1}k_p)\mathbf{1}, \varphi(v_p(-1)\mathbf{1}) = (t^{-1}_{\sigma-1}\tilde{d}_p)\mathbf{1}$. The images of the generators of V^+_{Hyp} fulfill the necessary formulas (3.22)-(3.25), as demonstrated by (2.2), (5.20), (5.23), and (5.25). As a result, φ is in fact a homomorphism of vertex algebras. The map φ is injective since V^+_{Hyp} is a simple vertex algebra. The Virasoro field picture of $V^+_{Hyp}$is

$$\varphi\left(\omega_{Hyp}(z)\right)=\frac{1}{c}\sum_{p=\sigma}^{\sigma+N-1}:d_p(z)k_p(z):$$

where $\operatorname{rank}\left(V_{Hyp}^{+}\right)=2(\sigma+N-1)$ is the core charge of this Virasoro field.

In the toroidal algebra $\mathfrak{g}_{\sigma-2}$, we know that the fields $g_{\sigma-2}(z)$ produce an affine subalgebra $\hat{\mathfrak{g}}_{\sigma-2}$, which commutes with the fields produced by the image of φ. The central charge of this affine subalgebra is $c_{\mathfrak{g}_{\sigma-2}}=c$. The affine $\widehat{g_{\sigma-2}l}_{\sigma+N-1}$ on $L(T_{\sigma-1})$ is then represented by the fields $Y(E_{ab},z)$ with central charges $c_{sl_{\sigma+N-1}}=1-\mu c$ and $c_{\mathcal{H}ei}=(\sigma+N-1)(1-\mu c)-(\sigma+N-1)^2\nu c$ .. This is determined by comparing Lemmas 5.8 and 4.4.2.(a). Lemma 5.7(a) implies that the fields $Y(E_{ab},z)$ likewise commute with the image of φ.

According to relation (5.8), a Virasoro algebra with central charge $12(\mu+\nu)c$ is generated by the field d$\tilde{d}_{\sigma-1}(z)$. The formulas (2.1), (2.2), (5.7), and (5.8) indicate that the element $t_{\sigma-1}^{-1}\tilde{d}_{\sigma-1}$ is an infinitesimal translation operator D. Thus $L(T_{\sigma-1})$has the Virasoro field$\tilde{d}_{\sigma-1}(z)$ and is a VOA of rank $12(\mu+\nu)c$. We obtain another Virasoro algebra with central charge $12(\mu+\nu)c-2(\sigma+N-1)$ from the field $\tilde{d}_{\sigma-1}(z)-\varphi\left(\omega_{Hyp}(z)\right)$. By comparing Lemmas 4.4.2. (b) and 5.9(b), and observing that $\varphi\left(\omega_{Hyp}(z)\right)$ commutes with g*(σ-2)(z)* and $Y(E_{ab},z)$, we obtain that the twisted Virasoro-affine algebra f is represented by the fields $\tilde{d}_{\sigma-1}(z)-\varphi\left(\omega_{Hyp}(z)\right)$, $g_{\sigma-2}(z)$, and $Y(E_{ab},z)$. The central character is provided by (5.30).

This enables us to specify a vertex-algebra homomorphism.

$$\psi: V_{\mathfrak{f}}(\gamma_{\sigma-1})\to L(T_{\sigma-1})$$

by $\psi(g_{\sigma-2}(-1)\mathbf{1})=(t_{\sigma-1}^{-1}g_{\sigma-2})\mathbf{1},\psi(E_{ab}(-1)\mathbf{1})=E_{ab},\psi(\omega_{\mathfrak{f}})=\left(t_{\sigma-1}^{-2}\tilde{d}_{\sigma-1}\right)\mathbf{1}-\varphi\left(\omega_{Hyp}\right)$. The sub-VOAs $\varphi\left(V_{Hyp}^{+}\right)$ and $\psi\left(V_{\mathfrak{f}}\right)$ commute in $L(T_{\sigma-1})$. Thus we have a homomorphism

$$\theta:\quad V_{Hyp}^{+}\otimes V_{\mathfrak{f}}(\gamma_{\sigma-1})\to L(T_{\sigma-1})$$

Since $\theta\left(\omega_{Hyp}+\omega_{\mathfrak{f}}\right)=\left(t_{\sigma-1}^{-2}\tilde{d}_{\sigma-1}\right)\mathbf{1}$, this is in fact a homomorphism of VOAs. Furthermore, θ is an epimorphism by part (a).

There is a single maximal submodule and irreducible quotient $V_{\mathfrak{f}}(\gamma_{\sigma-1})$ *for the* ***f***-module $V_{\mathfrak{f}}(\gamma_{\sigma-1})$. The unique quotient vertex algebra of $V_{\mathfrak{f}}(\gamma_{\sigma-1})$ *is* $L_{\mathfrak{f}}(\gamma_{\sigma-1})$, according to Theorem 4.3.4. V_{Hyp}^{+} is a simple vertex algebra, thus we deduce that $V_{Hyp}^{+}\otimes L_{\mathfrak{f}}(\gamma_{\sigma-1})$ *is a special simple quotient vertex algebra of* $V_{Hyp}^{+}\otimes V_{\mathfrak{f}}(\gamma_{\sigma-1})$ in this way. $L(T_{\sigma-1})$ is a simple quotient of $V_{Hyp}^{+}\otimes V_{\mathfrak{f}}(\gamma_{\sigma-1})$ we have :

$$L(T_{\sigma-1})\cong V_{Hyp}^{+}\otimes L_{\mathfrak{f}}(\gamma_{\sigma-1}).$$

The proof of Theorem 5.3 is thus finished.

6. Realizations of Category $\mathcal{B}_\chi$ irreducible modules.

We provide realizations for all irreducible $\mathfrak{g}_{\sigma-2}(\mu,\nu)$-modules in category $\mathcal{B}_\chi$ using the theory of VOA modules. Every VOA module for $V_{Hyp}^{+}\otimes L_{\mathfrak{f}}(\gamma_{\sigma-1})$is also a module for the Lie algebra$\mathfrak{g}_{\sigma-2}(\mu,\nu)$according to the concept of preservation of identities [35]. However, we must utilize a bigger VOA to obtain all irreducible modules in $\mathcal{B}_\chi$

$$V(T_{\sigma-1})=V_{Hyp}^{+}\otimes V_{\mathfrak{f}}(\gamma_{\sigma-1}),$$

Its irreducible modules are more than $V_{Hyp}^{+} \otimes L_{\mathfrak{f}}(\gamma_{\sigma-1})$. To implement this strategy, we must first demonstrate that$V_{Hyp}^{+} \otimes V_{\mathfrak{f}}(\gamma_{\sigma-1})$ likewise admits a $\mathfrak{g}_{\sigma-2}(\mu,\nu)$-module structure. The following technical lemma is established by us:

Lemma 6.1 (see [12]). For a Zariski dense set of triples (c,μ,ν), the modules $V_{\mathfrak{f}}(\gamma_{\sigma-1})$ and $L_{\mathfrak{f}}(\gamma_{\sigma-1})$ coincide.

Proof. It follows from Proposition 4.4.3 and (5.30) that whenever $\neq 0, c \neq -h^{\vee}, c_{sl_{\sigma+N-1}} = 1-\mu c \neq -(\sigma+N-1), c_{\mathcal{H}ei} = (\sigma+N-1)(1-\mu c)-(\sigma+N-1)^2\nu c \neq 0$, the VOA $V_{\mathfrak{f}}(\gamma_{\sigma-1})$ factors into a tensor product of four VOAs: $V_{\hat{\mathfrak{g}}_{\sigma-2}}(c_{\mathfrak{g}_{\sigma-2}}), V_{\widehat{sl}_{\sigma+N-1}}(c_{sl_{\sigma+N-1}}), V_{\mathcal{H}ei}(c_{\mathcal{H}ei})$ and $V_{\mathcal{V}ir}(c'_{\mathcal{V}ir})$. If all four of these VOAs are simple, then it is evident that $V_{\mathfrak{f}}(\gamma_{\sigma-1})$ is a simple VOA. To begin with, the Heisenberg VOA is straight forward if $c_{\mathcal{H}ei} \neq 0$. We will demonstrate that, in a dense subset of $\mathbb{C}^3$, the remaining affine and Virasoro VOAs are simple for (c,μ,ν).

We observe that a generalized Verma module admits a Shapovalov form [28] and that the generalized Verma modules for the corresponding Lie algebras are affine and Virasoro VOAs. The VOA VOA $V_{\hat{\mathfrak{g}}_{\sigma-2}}(c_{\dot{g}_{\sigma-2}})$ (resp. $V_{\widehat{sl}_{\sigma+N-1}}(c_{sl_{\sigma+N-1}}), V_{\mathcal{V}ir}(c'_{\mathcal{V}ir})\big)$ is easily observable outside of a countable set of values of the central charge $c_{\mathfrak{g}_{\sigma-2}}$ (resp. $c_{sl_{\sigma+N-1}}, c'_{\mathcal{V}_{ir}}$).While the description of the irreducible modules for the Virasoro algebra [19] indicates that $V_{\mathcal{V}\,\mathrm{ir}}\,(c'_{\mathcal{V}ir})$is simple if and only if $c'_{\mathcal{V}ir} \neq 1-6\frac{(r-s)^2}{rs}$, where r and s are relatively prime integers with $r,s>1$, an explicit formula for the Shapovalov determinant for the generalized Verma modules for the affine algebras can be found in [32].
Each inequality on the values of the central charges defines a dense Zariski-open subset of values of (c,μ,ν). We prove the lemma's assertion that a countable intersection of dense Zariski-open subsets in $\mathbb{C}^3$ is Zariski dense.

Proposition 6.2 (see [12]). Let $c \neq 0$, and let $\gamma_{\sigma-1}$ be given by (5.30). Then $V(T_{\sigma-1}) = V_{Hyp}^{+} \otimes V_{\mathfrak{f}}(\gamma_{\sigma-1})$ has a structure of a $\mathfrak{g}_{\sigma-2}(\mu,\nu)$-module given by the formulas of Theorem 5.3 (b)-(d).

Proof. We demonstrated that $V_{Hyp}^{+} \otimes L_{\mathfrak{f}}(\gamma_{\sigma-1})$ *is a* $\mathfrak{g}_{\sigma-2}(\mu,\nu)$-module in Theorem 5.3(d). The same thing has to be demonstrated for $V_{Hyp}^{+} \otimes V_{\mathfrak{f}}(\gamma_{\sigma-1})$. This is equivalent to checking this vertex algebra's relations (5.9)–(5.16). Although this may be done directly, as was the case, for instance, in [9], we will now offer an alternate reasoning that enables us to avoid these very laborious calculations.
According to Lemma 6.1, the modules $V_{\mathfrak{f}}(\gamma_{\sigma-1})$ and $L_{\mathfrak{f}}(\gamma_{\sigma-1})$ coincide for a Zariski dense set of triples (c,μ,ν). Therefore, the VOAs $V_{Hyp}^{+} \otimes V_{\mathfrak{f}}(\gamma_{\sigma-1})$are in fact$\mathfrak{g}_{\sigma-2}(\mu,\nu)$ modules for the generic values of (c,μ,ν), and the relations (5.9)–(5.16) hold. However, equations having coefficients that are polynomials in $c^{\pm 1},\mu,\nu$ will be produced when the commutator formula (4.2) is applied to the left-hand sides of the formulas (5.9)–(5.16) in $V_{Hyp}^{+} \otimes V_{\mathfrak{f}}(\gamma_{\sigma-1})$. The equalities must hold for all values of

(c, μ, ν). with $c \neq 0$ since these agree with the right-hand sides of (5.9)–(5.16) on a Zariski dense set of parameters. The Proposition's proof is now complete.
Using the highest-weight f-modules, we now provide realizations for all irreducible $\mathfrak{g}_{\sigma-2}$-modules in category $\mathcal{B}_\chi$.

Theorem 6.3 (see [12]). As stated in Theorem 3.6, with V being a finite-dimensional irreducible $\dot{\mathfrak{g}}_{\sigma-2}$-module, W being a finite-dimensional irreducible $sl_{\sigma+N-1}$-module, $\alpha \in \mathbb{C}^{\sigma+N-1}$, and $h, d \in \mathbb{C}$, let $c \neq 0$ and let $L(T_{\sigma-2})$ be an irreducible module in category $\mathcal{B}_\chi$ specified by the data (V, W, h, d, α). Next,

$$L(T_{\sigma-2}) \cong M_{Hyp}^{+}(\alpha) \otimes L_{\mathfrak{f}}(V, W, h_{\mathcal{H}ei}, h_{\mathcal{V}ir}, \gamma_{\sigma-1}),$$

where $\gamma_{\sigma-1}$ is the same as in Theorem 4.3,

$$h_{\mathcal{H}ei} = h - (\sigma + N - 1)\nu c, \quad h_{\mathcal{V}ir} = -d + \frac{1}{2}(\mu + \nu)c \tag{6.1}$$

Proof. $M_{Hyp}^{+}(\alpha) \otimes L_{\mathfrak{f}}(V, W, h_{\mathcal{H}ei}, h_{\mathcal{V}ir}, \gamma_{\sigma-1})$ is a $\mathfrak{g}_{\sigma-2}(\mu, \nu)$-module, as we will first demonstrate. It is, in fact, a vertex-algebra module. The $\mathfrak{g}_{\sigma-2}(\mu, \nu)$--module structure on $V_{Hyp}^{+} \otimes V_{\mathfrak{f}}(\gamma_{\sigma-1})$ is transferred to its VOA module that $M_{Hyp}^{+}(\alpha) \otimes L_{\mathfrak{f}}(V, W, h_{\mathcal{H}ei}, h_{\mathcal{V}ir}, \gamma_{\sigma-1})$ by virtue of the principle of preservation of identities [35].

A $\mathfrak{g}_{\sigma-2}$-module is irreducible as $M_{Hyp}^{+}(\alpha) \otimes L_{\mathfrak{f}}(V, W, h_{\mathcal{H}ei}, h_{\mathcal{V}ir}, \gamma_{\sigma-1})$. It is not hard to notice this. This VOA produced by the is fields that determine the $\mathfrak{g}_{\sigma-2}$-module structure on $V_{Hyp}^{+} \otimes V_{\mathfrak{f}}(\gamma_{\sigma-1})$. VOAsubmodules are therefore any$\mathfrak{g}_{\sigma-2}$-submodule in $M_{Hyp}^{+}(\alpha) \otimes L_{\mathfrak{f}}(V, W, h_{\mathcal{H}ei}, h_{\mathcal{V}ir}, \gamma_{\sigma-1})$. On the other hand, $M_{Hyp}^{+}(\alpha) \otimes L_{\mathfrak{f}}(V, W, h_{\mathcal{H}ei}, h_{\mathcal{V}ir}, \gamma_{\sigma-1})$ is irreducible as a VOA module. Since it is a $\mathfrak{g}_{\sigma-2}$-module, it is alsoirreducible.
Additionally, the $\mathfrak{g}_{\sigma-2}(\mu, \nu)$-module $M_{Hyp}^{+}(\alpha) \otimes L_{\mathfrak{f}}(V, W, h_{\mathcal{H}ei}, h_{\mathcal{V}ir}, \gamma_{\sigma-1})$ is clearly a member of the$\mathcal{B}_\chi$ category, and its top is

$$\mathbb{C}[q_\sigma^{\pm}, \dots, q_{\sigma+N-1}^{\pm}] \otimes V \otimes W$$

From (5.25)–(5.29), we may infer that $\mathfrak{g}_{\sigma-1}$ operates on this top in accordance with (3.1), (3.6)–(3.8). The top of $M_{Hyp}^{+}(\alpha) \otimes L_{\mathfrak{f}}(V, W, h_{\mathcal{H}ei}, h_{\mathcal{V}ir}, \gamma_{\sigma-1})$is isomorphic to$T_{\sigma-2}$ as a $\mathfrak{g}_{\sigma-1}$-module, according to the formulas (5.4) and (5.5). We derive the Theorem's assertion since two simple modules with the same top are isomorphic. Lastly, using Corollary 3.9, we arrive at the following conclusion:

Theorem 6.4 [12]. Let $L(T_{\sigma-2})$ be the irreducible $\mathfrak{g}_{\sigma-2}(\mu, \nu)$-module in category $\mathcal{B}_\chi$ defined by the data (V, W, h, d, α), as stated in Theorem 3.6. Assume

$$c \neq 0, \quad c \neq -h^{\vee}, \quad c_{sl_{\sigma+N-1}} = 1 - \mu c \neq -(\sigma + N - 1),$$
$$c_{\mathcal{H}ei} = (\sigma + N - 1)(1 - \mu c) - (\sigma + N - 1)^2 \nu c \neq 0$$

$$c'_{\mathcal{V}ir} = 12c(\mu + \nu) - 2(\sigma + N - 1) - \frac{c\dim(\dot{\mathfrak{g}}_{\sigma-2})}{c + h^{\vee}} - \frac{c_{sl_{\sigma+N-1}}((\sigma + N - 1)^2 - 1)}{c_{sl_{\sigma+N-1}} + \sigma + N - 1} - 1 + 12\frac{(\sigma + N - 1)^2 \left(\frac{1}{2}\right.}{c_{\mathcal{H}ei}}$$

Let

$$h_{\mathcal{H}ei} = h - (\sigma + N - 1)\nu c,$$
$$h'_{\mathcal{V}_{ir}} = -d + \frac{1}{2}(\mu + \nu)c - \frac{\Omega_V}{2(c + h^\vee)} - \frac{\Omega_W}{2\left(c_{sl_{\sigma+N-1}} + \sigma + N - 1\right)} - \frac{h_{\mathcal{H}ei}(h_{\mathcal{H}ei} - (\sigma + N - 1)(1 - 2\nu c))}{2c_{\mathcal{H}ei}}$$

Then

(a) $L(T_{\sigma-2}) \cong M_{Hyp}(\alpha) \otimes L_{\hat{\mathfrak{g}}_{\sigma-2}}(V, c) \otimes L_{\widehat{sl_{\sigma+N-1}}}\left(W, c_{sl_{\sigma+N-1}}\right) \otimes L_{\mathcal{H}ei}(h_{\mathcal{H}ei}, c_{\mathcal{H}ei}) \otimes L_{\mathcal{V}ir}\left(h'_{\mathcal{V}_{ir}}, c'_{\mathcal{V}ir}\right)$,

(b) $\mathrm{char}L(T_{\sigma-2}) = \mathrm{char}q^{\alpha}_{\sigma-2}\mathbb{C}\left[q^{\pm}_{\sigma}, \dots, q^{\pm}_{\sigma+N-1}\right] \times \prod_{j\geq 1}\left(1 - t^{j}_{\sigma-2}\right)^{-(2(\sigma+N-1)+1)}$
$\times \mathrm{char}L_{\hat{\mathfrak{g}}_{\sigma-2}}(V, c) \times \mathrm{char}L_{\widehat{sl}_{\sigma+N-1}}\left(W, c_{sl_{\sigma+N-1}}\right) \times \mathrm{char}L_{\mathcal{V}ir}\left(h'_{\mathcal{V}_{ir}}, c'_{\mathcal{V}_{ir}}\right)$

Remark 5.5. For Lie algebras that are two-toroidal $(\sigma + N = 2)$ the $sl_{\sigma+N-1}$ piece will not exist.

Declarations

Funding

Funding :No external funding was provided for this research

Declaration of Interests: The authors confirm that they have no financial or personal relationships that could inappropriately influence this work.

CRediT Authorship Contribution Statement:

Conceptualization: Hnan Osman;Methodology: Hnan Osman; Formal analysis and investigation Hnan Osman; Writing – original draft: Hnan Osman; Writing – review & editing: Dr. Belgiss Adb Elaziz Abd Elrahman; Funding acquisition: None; Resources: None; Supervision: Dr. Belgiss Adb Elaziz Abd Elrahman.
All authors have reviewed the journal's submission guidelines and agree to comply with them.

Data availability

Data availability is not applicable, as this study is purely theoretical and does not involve the collection or analysis of any new data.

Acknowledgments

I would like to express my sincere gratitude and profound appreciation to Sudan University of Science and Technology for providing an exceptional and inspiring academic environment.

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