Innovations

Stability Analysis of Periodic Solutions of Vander Pol- Duffing Oscillator with Asymmetric Nonlinearities

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Abstract
This paper investigated the stability of periodic solution of Vander Pol-Duffing forced Oscillator with asymmetric nonlinearities using Lyaponov second method. Through the exploits of the critical points of the system, the solution of the oscillator was found to be periodic. Three equilibra points of the system were obtained due to the quadratic and cubic nature of the oscillator. A suitable and complete Lyaponov functional of the model were considered using the total energy of the system. The existence of negative definite time derivative of the Lyaponov function confirmed the asymptotic stability of the equilibrium point of the system. The other two equilibrium points showed one region of instability and the region of stability. Furthermore, Mathcad software was used to analyze the numerical behavior of the system which improved and extended some known results in literature.

Keywords: *Stability, Lyaponov method, Vander Pol-Duffing equations, Asymmetric non linearites*

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1: Introduction

This paper is motivated by the works of [1,2,21, 30, 31] and the aim is to investigate the stability analysis of periodic solutions of Vander Pol –Duffing oscillator with asymmetric non linearities. Consider the Vander Pol –Duffing oscillator of the form

$$\ddot{x}-\mu\left(1-x^2\right)\dot{x}+\alpha^2x+\beta x^3=Fcos\left(wt\right)$$

(1.1)

where $x=x(t)$ denotes displacement at time t, $\dot{x}$ denotes the derivative of x with respect to t i.e. velocity, $\ddot{x}$ denotes the second derivative with respect to t i.e. acceleration, μ denotes the damping parameter, α and β are constant that control the amount of nonlinearity in the restoring force, w denotes the frequency and F denotes the amplitude of the external force. Equation (1.1) was first explored and reported by [28]. When $\mu=0$ equation (1.1) becomes Duffing equation which is given by

$$\ddot{x}+\alpha^2x+\beta x^3=Fcos\left(wt\right)$$

(1.2)
Equation (1.2) was investigated by Duffing when he was working on vibrations. He examined the effect of quadratic and cubic stiffness nonlinearities. When $\beta = 0$ equation (1.1) becomes Vander Pol equation which is given by

$$\ddot{x} - \mu\left(1 - x^2\right)\dot{x} + \alpha^2 x = Fcos\left(wt\right)$$

(1.3)
Vander Pol was proposed by Dutch electrical engineer and physicist Balthasar van der Pol while he was studying electrical circuits. When $\mu = 2$ and $\beta = 4$ equation (1.1) is given by

$$\ddot{x} - 2\left(1 - x^2\right)\dot{x} + \alpha^2 x + 4x^3 = Fcos\left(wt\right)$$

(1.4)
with initial conditions $x\left(0\right) = x\left(2\pi\right)$ and $\dot{x}\left(0\right) = \dot{x}\left(2\pi\right)$ where x denotes the displacement with time $\ddot{x}$ denotes the derivative t with respect to x.

$\alpha^2 x^+ + 43x^{3+}$ of (1.4) is asymmetric nonlinearity which is both quadratic and cubic and positive overall and very close to nonlinear resonance.

Due to appearance of asymmetric nonlinearity the methods of solutions employed in [21, 30, 34] are no longer valid and the damping coefficient is negative. Lyaponov function will be constructed and theorem of Lyaponov will take care of the problems of asymmetric nonlinearity in order to achieve the required stability.

The aim of this paper to investigate the stability analysis of VanderPol-Duffing oscillator with asymmetric nonlinearity from equation (1.4). We constructed a Lyaponov function and usedsome exploits on the parameters to overcome the issue of asymmetric nonlinearities.

The Vander Pol – Duffing oscillator is a second order nonlinear differential equation that is employed to solve physical, engineering and even biological problems. This oscillator type is a generalization of the classic Vander- Pol oscillator. The oscillator has rich dynamic behavior including chaos and bifurcation [29]. The oscillator has a wide spread of applications in the modeling of nonlinear oscillation processes. For example it is used to model optical instability in a dispersive medium in which the refractive index is dependent on the optical intensity [18].

Under certain parabolic conditions [12] obtained the first integral of Duffing – Vander – type oscillator using the method of differentiable dynamics and also analyzed the equilibrium points. Lie symmetry reduction method was further used to find two nontrivial infinitesimal generators where variables were constructed [10].

In [13], a novel phenomenological model of EEG signal was proposed which is based on the dynamics of a coupled Van-der-pol Duffing oscillator network and some interesting phenomena. [22]investigated the phenomena of chaos synchronization and efficient signal transmission in a physically interesting model named the Vander Pol-Duffing oscillator.

During the last decade, many scholars have used different techniques to investigate Vander Pol Duffing oscillator. For instance, see[28], [24], [19], [27],[14], [1], [26], [5] and [32].

Stability is one of the analytical properties of solutions of differential equations. A system might be stablewithout being asymptotical stable but asymptotic stability will always imply stability. See [7],[8] and [9]. Stability describes the behavior of a system when the system undergoes small changes and is determined by the interval placed on the time derivative of the Lyapunov candidate that is formed using thedifferential equation [9].

Many authors have worked on the stability analysis of periodic solution of Vander Pol-Duffing forced oscillator using different techniques. See for instance [14], [36], [16], [2] and [14].

The advantages of using Lyponov theory and functional in the analysis of periodic solutions of Vander Pol-Duffing oscillator cannot be over emphasized. The theory is periodically necessary when dealing with uncertain systems and nonlinear limit – varying parameters.
The theory possesses techniques that are effective and insightful. The functional allows us to assess the stability and instability of a system without solving the differential equation that describes the system and serves as a tool in the control of a dynamical system.
The applications of oscillators which are mechanical devices that produce a periodic oscillator between two bodies based on changes in energy are in the areas of seismology, physics, economics etc. They are used in weak signal detection [20]. Also widely used in signal communication domain such as in the secure communication field [33]. They also can be applied in marine such as ship propeller, blade number recognition [31], [24] and [23].
The paper is arranged as follows, in section 2 we discussed the preliminaries, and section 3 discussed on results and in section 4 we took the numerical simulation of the behavior and section 5 is about conclusions.

2: Preliminaries

Definition 2.1 Consider a real valued function $V : \mathrm{R}^n \to \mathrm{R}$ which is continuously differentiable with $V(0)=0$. Then the function is said to be

(i) Positive definite if $V(x)>0$ for all $x \neq 0$.

(ii) Negative definite if $V(x)<0$ for all $x \neq 0$.

(iii) Positive semi-definite if $V(x)\geq 0$ and can vanish for some $x \neq 0$.

(iv) Negative semi-definite if $V(x)\leq 0$ and can vanish for some $x \neq 0$.

Definition 2.2 (Stability in the sense of Lyapunov): Consider a nonlinear system

$$\dot{x} = f(x,t),\, x(t_0) = x_0$$

(2.1)

where $x(t) \in D \subseteq \mathrm{R}^n$ denotes the system state vector, D is an open set containing the origin and $f : D \to \mathrm{R}^n$ is continuous on D. Assuming f has an equilibrium at x_e so that $f(x)=0$ then equation (3.1) is said to be stable in the sense of Lyapunov if for any $\varepsilon > 0$ there exist $\delta = \delta(\varepsilon, t_0) > 0$ such that $x(0)-x_e < \delta \Rightarrow x(t)-x_e < \varepsilon$ for all $t \geq 0$. Equation (2.1) is said to be uniform if $\delta = \delta(\varepsilon)$ that is δ is independent of t_0 over the entire time interval.

Definition 2.3 (Asymptotic Stability): Equation (2.1) is said to be asymptotically stable if it is Lyapunov stable and there exist $\delta = \delta(t_0) > 0$ such that $x(0)-x_e < \delta \Rightarrow \lim_{t\to\infty} x(t)-x_e = 0$. Thus asymptotic stability means that solutions which start close enough will not only remain close enough but also converge to the equilibrium as time becomes infinite.

Theorem 2.4 Consider the autonomous system

$$\frac{dx}{dt} = F(x)$$

(2.2)

with a critical point at the origin. If there exist a function $V(x)$ that is
(i) Continuous and has continuous first partial derivatives.
(ii) Positive definite about the origin. Then if

$$\frac{dv}{dt}(x) := \frac{\partial v}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial v}{\partial x_2}\frac{dx_2}{dt} + \ . \ . \ . + \frac{\partial v}{\partial x_n}\frac{dx_n}{dt} = \nabla V.F(x)$$

is negative semi-definite about the origin, then the origin is a stable critical point for equation (2.2). If $\frac{dv}{dt}(x) < 0$ for all x within a small ball about the origin, then the origin is an asymptotically stable critical point for equation (2.2).

Theorem 2.5: Let the origin be an isolated critical point for equation (2.2) and let $V(x)$ be a function that is continuous and has continuous first partial derivatives. Suppose $V(0,0)=0$ and every neighborhood of the origin has at least one point at which V is positive and there is a domain D containing the origin such that $\nabla V.F(x)$ is positive definite, then the origin is an unstable critical point.

Theorem 2.6: Suppose there exist a function $V: \mathrm{R}^n \to \mathrm{R}$ that is continuously differentiable and satisfies the following conditions
(i) $V(x)$ is positive definite.
(ii) The time derivative $\dot{V}$ of $V(x)$ along the solution path is negative definite. Then the trivial solution $x=0$ is asymptotically stable.

Definition 2.7 (Periodic Solution): Periodic solutions are solutions that describe regular repeating processes. It is the solution $y = f(x)$ of an equation with property that there exists a positive real number $k \neq 0$ such that $f(x+k) = f(x)$. k is called the period of the function. If a periodic function has a period k, then $f(x+k) = f(x+2k) = f(x)$. In general $(x+nk) = f(x)$, $k \in \mathrm{Z}$, $n \in \mathrm{N}$ and for all $x \in$. A periodic function repeats its values at regular intervals.

Definition 2.8 (Stability): An equilibrium solution f_e to an autonomous system is said to be

(a) Stable if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that every solution $f(t)$ having initial condition within distance δ $i.e \| f(t_0) - f_e \| < \delta$ of the equilibrium remains within the distance ε i.e. $\| f(t) - f_e \| < \varepsilon$ for all $t \geq t_0$

(b) Asymptotically stable if it is stable and in addition, there exists $\delta_0 > 0$ such that whenever $\| f(t_0) - f_e \| < \delta_0$ then $f(t) \to f_e$ as $t \to \infty$

Definition 2.9 (Limit Cycle): A limit cycle is a closed trajectory in phase space having the property that at least one other trajectory spirals into it either as time approaches infinity.

Definition 2.10 (Oscillator): An oscillator is a mechanical or electronic device that works on the principles of oscillation.

Definition 2.11 (Forced Oscillator):A forced oscillator is one in which a force is continually or repeatedly applied to an oscillating system to keep the oscillation in motion.

Definition 2.12 (Vander Pol Equations): It is an equation describing self-sustaining oscillations in which energy is fed into small oscillations and removed from large oscillations.

3: Results and Discussion

Consider the nonlinear differential equation

$$\ddot{x} - \mu\left(1-x^2\right)\dot{x} + \alpha^2 x + \beta x^3 = Fcos\left(wt\right)$$

(3.1)

when $\mu = 2$ and $\beta = 4$ equation (3.1) describes the soft spring form of Van der Pol-Duffing oscillator given by

$$\ddot{x} - 2\left(1-x^2\right)\dot{x} + \alpha^2 x + 4x^3 = Fcos\left(wt\right)$$

(3.2)

In order to find the equilibrium points of equation (3.2) we assume that $F = 0$. Hence equation (3.2) becomes

$$\ddot{x} - 2\left(1-x^2\right)\dot{x} + \alpha^2 x + 4x^3 = 0$$

(3.3)

which is an unforced form of Van der Pol-Duffing equation. Let $x = x_1$ then equation (3.3) becomes

$$\ddot{x}_1 - 2\left(1-x_1^2\right)\dot{x}_1 + \alpha^2 x_1 + 4x_1^3 = 0$$

(3.4)

Let $\dot{x}_1 = x_2$ and $\ddot{x}_1 = \dot{x}_2$. Then equation (3.4) becomes

$$\dot{x}_2 - 2\left(1-x_1^2\right)x_2 + \alpha^2 x_1 + 4x_1^3 = 0$$

$$\dot{x}_2 = 2\left(1-x_1^2\right)x_2 - \alpha^2 x_1 - 4x_1^3$$

(3.5)

Hence the equivalent system is thus

$$\left.\begin{aligned} \dot{x}_1 &= 0x_1 + x_2 \\ \dot{x}_2 &= 2\left(1-x_1^2\right)x_2 - \alpha^2 x_1 - 4x_1^3 \end{aligned}\right\}$$

(3.6)

At stationary point $\dot{x}_1 = 0$ and $\dot{x}_2 = 0$. When $\dot{x}_1 = 0$ then $x_2 = 0$. Hence

$$0 = -\alpha^2 x_1 - 4x_1^3$$

(3.7)

Multiplying equation (3.7) by -1, we have

$$\alpha^2 x_1 + 4x_1^3 = 0$$

$$x_1\left(\alpha^2 + 4x_1^2\right) = 0 \Rightarrow x_1 = 0 \text{ or } \alpha^2 + 4x_1^2 = 0$$

(3.8)

For $\alpha^2 + 4x_1^2 = 0$, we have

$$x_1 = \frac{-0 \pm \sqrt{0^2 - 4\times 4\times \alpha^2}}{2\times 4}$$

where $a=4,\ b=0$ and $c=\alpha^2$

$$x_1 = \frac{\pm\sqrt{-16\alpha^2}}{8}$$

(3.9)

which under further simplification gives $x_1 = \pm\frac{\alpha i}{2}$. The roots of the equation are $\frac{\alpha i}{2}$ and $-\frac{\alpha i}{2}$ with the general solution

$$x_1(t) = A cos\frac{\alpha}{2}t + B sin\frac{\alpha}{2}t$$

(3.10)

Equation (3.10) shows that the solution of Van der Pol-Duffing forced oscillator is periodic. The equilibrium points of the system are $(0,0)$, $\left(\frac{\alpha}{2}i,0\right)$ and $\left(-\frac{\alpha}{2}i,0\right)$. However, since there is no external force acting on the system, we know that the total energy of the system is constant. Hence the total energy is given by

Total energy =Kinetic energy +Potential energy

$$E = \frac{1}{2}\dot{x}_1^2 + \int_0^{x_1} f(x_1)dx_1$$

(3.11)

where $f(x_1) = \alpha^2 x_1 + 4x_1^3$

$$\int_0^{x_1}(\alpha^2 x_1 + 4x_1^3)dx_1 = \frac{\alpha^2 x_1^2}{2} + \frac{4x_1^4}{4} = \frac{\alpha^2 x_1^2}{2} + x_1^4$$

(3.12)

Hence the total energy is given by

$$E = V(x_1, x_2) = \frac{1}{2}\dot{x}_1^2 + \frac{\alpha^2 x_1^2}{2} + x_1^4$$
$$= \frac{1}{2}x_2^2 + \frac{\alpha^2 x_1^2}{2} + x_1^4$$

(3.14)

Equation (3.14) is the Lyapunov function of equation (3.6) which depends on the stiffness constant called α. It is also a potential function whose values are a physical potential. For stability test we have

(i) $V(x_1, x_2) = \frac{1}{2}x_2^2 + \frac{\alpha^2 x_1^2}{2} + x_1^4 > 0$

(3.15)

therefore $V(x_1, x_2)$ is positive definite for all $(x_1, x_2) \neq 0$

(ii) $V(0,0) = 0$

(3.16)

(iii) $\dot{V}(x_1, x_2) = \frac{\partial v}{\partial x_1} \cdot \frac{dx_1}{dt} + \frac{\partial v}{\partial x_2} \cdot \frac{dx_2}{dt}$

(3.17)

$$= \left(\alpha^2 x_1 + 4x_1^3\right)x_2 + x_2\left(2x_2 - 2x_1^2 x_2 - \alpha^2 x_1 - 4x_1^3\right)$$
$$= \alpha^2 x_1 x_2 + 4x_1^3 x_2 + 2x_2^2 - 2x_1^2 x_2^2 - \alpha^2 x_1 x_2 - 4x_1^3 x_2$$
$$= -2x_1^2 x_2^2 + 2x_2^2 = -2x_2^2\left(x_1^2 - 1\right) < 0$$

(3.18)

Equation (3.18) is negative definite. By theorem (3.6) the equilibrium point is asymptotically stable and we conclude that the system is stable and hence asymptotically stable.

4: Numerical Simulation of the Results

We illustrate the numerical solution of our results in (4.1) using the MATHCAD software;

Simulation-1

$\mu := 2,\ \beta := 2,\ \alpha := 4,\ \omega := 1,\ F := 1$

Define a function that determines a vector of derivative value at any solution point (t, X)

$$D(t, X) := \begin{bmatrix} X_1 \\ F\cos(\omega t) - \mu\left(1 - (X_0)^2\right)X_1 - \beta^2 X_0 - \alpha(X_0)^3 \end{bmatrix}$$

$T1 = 50$ Endpoint of solution interval

Define additional argument for the ODE solver

$t0 := 0$ Initial value of independent variable

$t1 := 10$ Final value of independent variable

$X_0 := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ Vector of initial function values

$num := 1 \times 10^3$ Number of solution values on $[t0, t1]$

Solution matrix

$S1 :=$ Rkadapt $(X_0, t0, t1, num, D)$

$x := S1^{0}$ Independent variable values

$x1 := S1^{1}$ First solution function values

$x2 := S1^{2}$ Second solution function values

Table 1: Solution matrix table when $F = 1$

	0	1	2
0	0	0	1
1	0.01	$9.95 \cdot 10^{-3}$	0.99
2	0.02	0.02	0.98
3	0.03	0.03	0.969
4	0.04	0.039	0.959
5	0.05	0.049	0.948
6	0.06	0.058	0.937
7	0.07	0.067	0.926
8	0.08	0.077	0.914
9	0.09	0.086	0.903
10	0.1	0.095	0.892
11	0.11	0.104	0.88
12	0.12	0.112	0.868
13	0.13	0.121	0.856
14	0.14	0.129	0.845
15	0.15	0.138	...

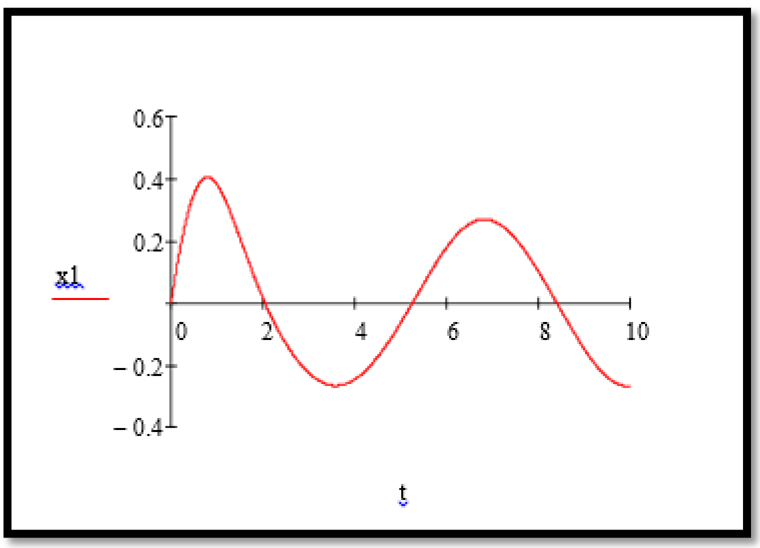

Figure1: Trajectory profile of the Vander Pol Duffing forced oscillator which starts out from the origin and moves away from the equilibrium point on the positive axis. The system is periodic with unstable equilibrium point.

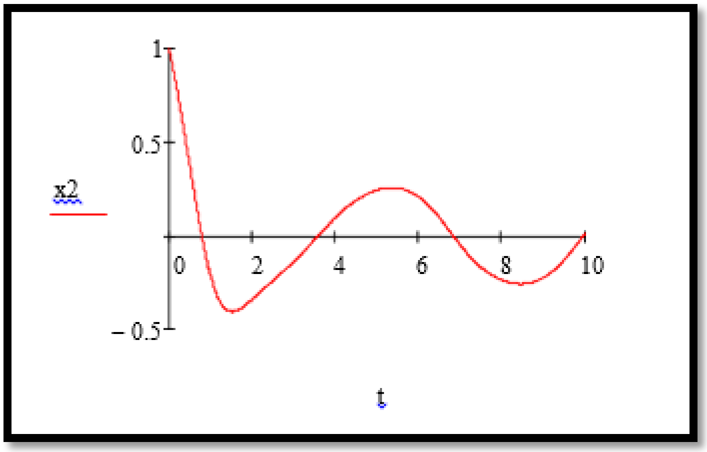

Figure 2: Trajectory profile of Vander Pol Duffing forced oscillator showing unstable nature of the equilibrium point.

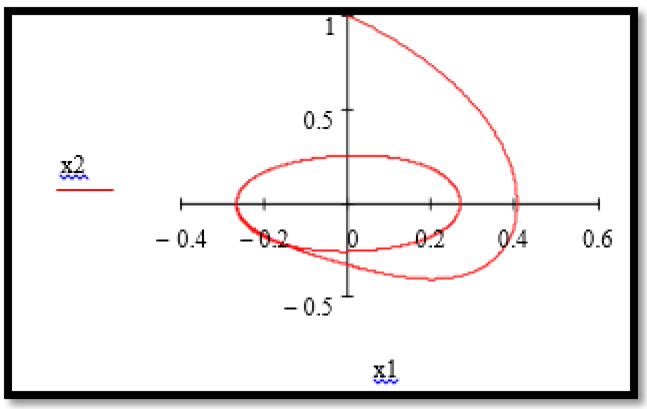

Figure 3: Phase portrait of Vander Pol Duffing forced oscillator depicting asymptotic stability of the equilibrium point.

Simulation 2

$\mu{:=}1,\ \beta{:=}1,\ \alpha{:=}2,\ \omega{:=}1,\ F{:=}2$

S1 =

	0	1	2
0	0	0	1
1	0.01	0.01	1.01
2	0.02	0.02	1.02
3	0.03	0.03	1.029
4	0.04	0.041	1.038
5	0.05	0.051	1.048
6	0.06	0.062	1.056
7	0.07	0.072	1.065
8	0.08	0.083	1.074
9	0.09	0.094	1.082
10	0.1	0.105	1.09
11	0.11	0.116	1.098
12	0.12	0.127	1.106
13	0.13	0.138	1.113
14	0.14	0.149	1.121
15	0.15	0.16	...

Table 2: Solution matrix table for $F=2$

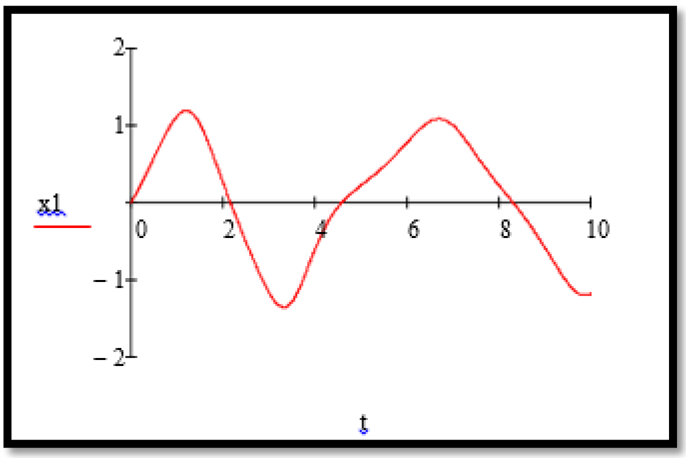

Figure 4: Trajectory profile of Vander Pol Duffing forced oscillator showing instability of the equilibrium point. This is as a result of the forcing term present in the given model.

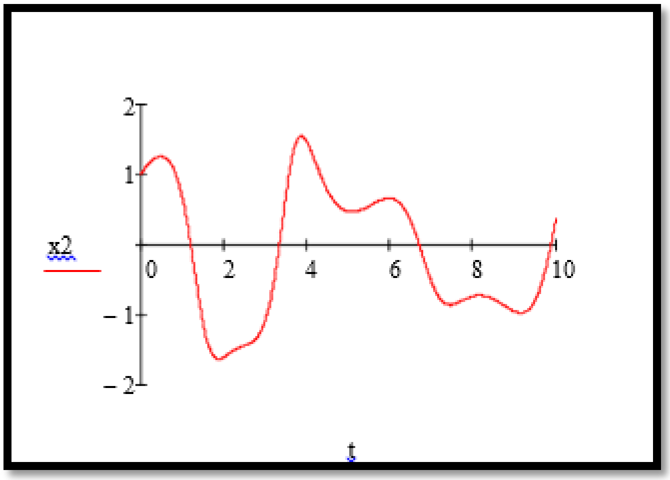

Figure 5: Trajectory profile of Vander Pol Duffing forced oscillator showing mutations, aperiodic and unstable nature of the equilibrium point.

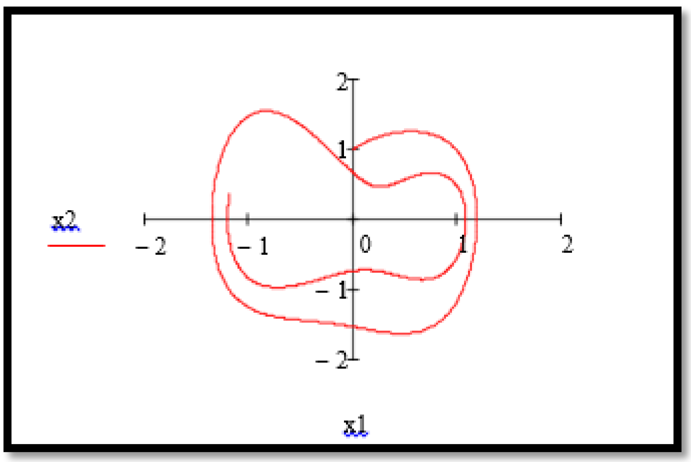

Figure 6: Phase portrait of Vander Pol Duffing forced oscillation depicting a non-conservative system that is not periodic and unstable in nature.

Simulation 3

$\mu{:=}2,\ \beta{:=}2,\ \alpha{:=}4,\ \omega{:=}1,\ F{:=}0$

S1 =

	0	1	2
0	0	0	1
1	0.01	$9.9 \cdot 10^{-3}$	0.98
2	0.02	0.02	0.96
3	0.03	0.029	0.94
4	0.04	0.038	0.92
5	0.05	0.048	0.9
6	0.06	0.056	0.88
7	0.07	0.065	0.861
8	0.08	0.074	0.841
9	0.09	0.082	0.821
10	0.1	0.09	0.802
11	0.11	0.098	0.782
12	0.12	0.106	0.763
13	0.13	0.113	0.743
14	0.14	0.121	0.724
15	0.15	0.128	...

Table 3: Solution matrix table for $F = 0$

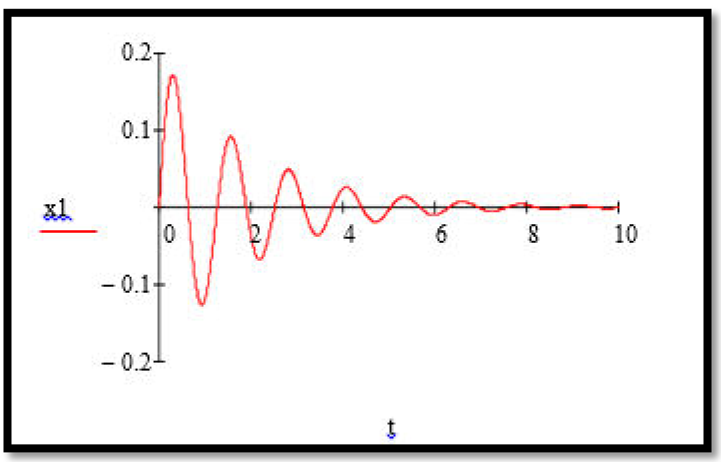

Figure 7: Trajectory profile of Vander Pol Duffing forced oscillator which is unstable and periodic. The oscillatory nature reduces as the trajectory moves away from the equilibrium point.

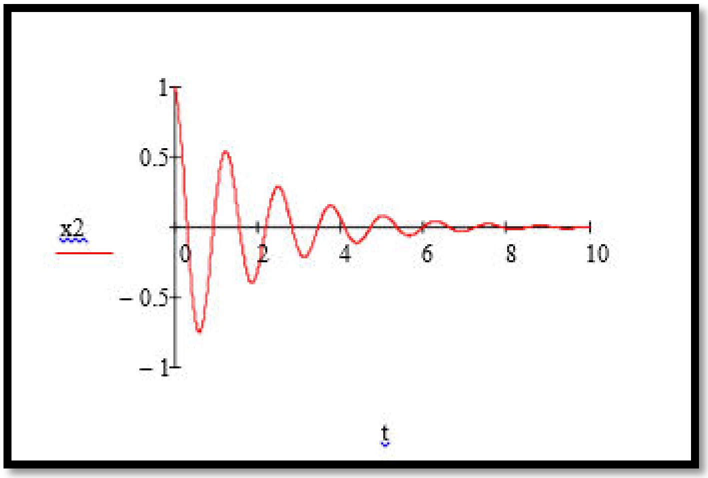

Figure 8: Trajectory profile of Vander Pol Duffing forced oscillator showing periodic and unstable nature of the system.

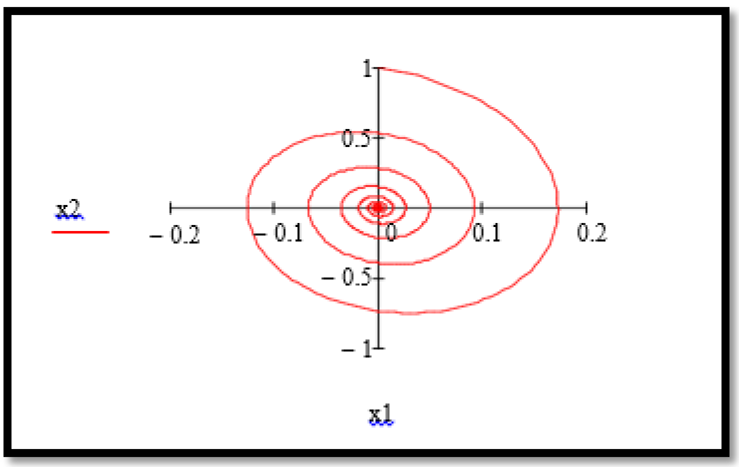

Figure 9: Phase portrait of Vander Pol Duffing forced oscillator depicting asymptotic stability of the equilibrium point.

5: Conclusions

From the results of this work, the study of the stability analysis of periodic solution using the Lyapunov second method. We discovered that;

(i) At the stationary point, the solution of the system is periodic which depends on the stiffness constant α.

(ii) Three equilibria points were achieved which are $(0,0)$, $\left(-\frac{\alpha}{2}i,0\right)$ and $\left(\frac{\alpha}{2}i,0\right)$. At $(0,0)$, the system is asymptotically stable. At $\left(-\frac{\alpha}{2}i,0\right)$, the system is unstable. At $\left(\frac{\alpha}{2}i,0\right)$, the system is unstable.

(iii) The equilibrium point is asymptotically stable. The result was achieved using the Lyapunov second method in which Lyapunov function $V(x_1,x_2)>0$ and $\dot{V}(x_1,x_2)<0$.

(iv)The numerical behavior of the system was demonstrated using Mathcad. The Mathcad showed different trajectory describing stability and instability regions of the system. The numerical behaviors are explained as follows:

In Figure 1, thetrajectory profile of the Vander Pol Duffing forced oscillator which started out from the origin and moved away from the equilibrium point on the positive axis showed the system to be periodic with unstable equilibrium point.

In Figure 2, the trajectory profile of Vander Pol Duffing forced oscillator showed unstable nature of the equilibrium point.

In Figure 3, thePhase portrait of Vander Pol Duffing forced oscillator depicted asymptotic stability of the equilibrium point.

In Figure 4, the trajectory profile of Vander Pol Duffing forced oscillator showed instability of the equilibrium point. This was as a result of the forcing term present in the given model.

In Figure 5, the trajectory profile of Vander Pol Duffing forced oscillator showed mutations, aperiodic and unstable nature of the equilibrium point.

In Figure 6,the Phase portrait of Vander Pol Duffing forced oscillation depicted a non-conservative system that is not periodic and unstable in nature.

In Figure 7 the trajectory profile of Vander Pol Duffing forced oscillator was shown for $\mu:=2$, $\beta:=2$, $\alpha:=4$, $\omega:=1$, $F:=0$ which is unstable and periodic. The oscillatory nature reduces as the trajectory moves away from the equilibrium point.

In Figure 8, the trajectory of Vander Pol Duffing forced oscillator was parallel to $t-$axis showing the non-periodic and unstable nature of the system.

In Figure 9, a phase portrait of Vander Pol Duffing forced oscillator was shown. The Phase portrait of shows that the equilibrium point is asymptotically stable

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