Innovations

Stability and Hopf Bifurcation Analysis of Periodic Solutions of a Duffing Equation

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Abstract
In this study, the stability and Hopf bifurcation analysis of periodic solution of Duffing Equation were considered. Also other type of bifurcation like the saddle-node, trans-critical and pitch fork were also studied. The eigen value, Jacobian and Floquet theory were used to analyse both the stability and Hopf bifurcation of the periodic solutions of the equilibrium points. The result showed that equilibria points have at most three T-periodic solutions under a strong damped conditions due to the cubic non-linearities. The bifurcation points showed one critical and another subcritical.
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Introduction
Consider the Duffing Oscillator described by the Differential equation
$\ddot{x} + \alpha\dot{x} - \tau^2 x + \beta x^3 = f\cos\omega t$ ……. $f \geq 0$
(1.1)
with initial conditions

$$x(0) = x(2\pi)$$
$$\dot{x}(0) = \dot{x}(0)$$

(1.2)
Where $\alpha, \beta, \tau^2, \omega, t$, f and $x$represents damping coefficient, nonlinear (cubic) stiffness parameter, the linear stiffness parameter, the frequency, the independent variable, the strength of the driving force and the displacement respectively, which differ from the elementary example of a forced and damped harmonic oscillator ($\ddot{x} + \alpha\dot{x} - \tau^2 x = f\cos\omega t$) only by the nonlinear terms βx^3, which changes the dynamics of the system drastically.
The Duffing equation is a nonlinear, non-autonomous equation introduced by [3] as a dynamical equation that exhibit chaotic behaviour and as a harmonic oscillator modified by a cubic nonlinearity and driven harmonically. This equation describe the motion of a damped oscillator with more complex potential than in simple harmonic motion. From mathematical point of view, this equation is second nonlinear equation, the equation is used to model"hard and soft spring" and the dynamic of a point mass in double well potential and can be regarded as a periodic forced steel beam which is deflected towards the two magnets.

Moreso, its application are seen in the modelling of variety of physical processes in mechanical and electrical engineering and in physics which describes the soliton solution to important physics models such as the Kdv equation, mKdv equation, sine-Gordon, non-linear Schrodinger equation and shallow water wave equation.

Due to the wide range of application of the Duffing equation in different field of endeavour, Many researchers have worked on the Duffing equation using different methods which yields amazing results. See [1],[2], [3], [5], [17], [19] and [21].

Stability is a qualitative property for Differential Equations that is crucial in both linear and nonlinear analysis. It describes the behaviour of the system when the system undergo small changes. Analytically, stability is determined by interval placed on the given Differential Equation. For linear system with one equilibrium point, but for nonlinear system with more than two parameters, the search for the stability and analysis of the equilibrium point becomes a challenge.

Empasis on stability has been researched on and discussed by many authors. For instance see [2], [5], [6], [7], [8] and [14] and their reference therein. For other researchers who have worked on the stability of non-linear systems, see [17], [18], [19], [20], [22], [23] and [24]

Motivated by the above literature, the goal of this paper is to investigate the stability analysis of periodic solution of Duffing oscillator. The paper further investigated the Hopf bifurcation analysis of the periodic solution of the Duffing Oscillator and the behaviour of the system was analysed using the MATHCAD software.

2. Preliminaries

Bifurcation theory

A bifurcation occurs when a small smooth change made to the parameter values (the bifurcation parameter) of a system causes a sudden "qualitative" or topological change in behaviour. Generally, at a bifurcation, the local stability properties of equilibria, periodic orbits or other variant sets changes.[4].

Bifurcation of dimension one

Consider the scalar differential equations of the form

$$\frac{du}{dt} = f(u, \mu)$$

2.1

Where u is a real valued function of the the time t, the vector field f is real valued depending, besides u, upon a parameter μ. The parameter μ is the bifurcation parameter. We suppose that equation 2.1 is well posed and satisfies the hypothesis of the Cauchy-Lipschitz theory, such that for each initial condition there exist a unique solution of (2.1). Furthermore we assume that the vector field is of class $C^k, k \geq 2$, in a neighbourhood of (0,0) satisfying

$$f(0,0) = 0, \frac{\partial f}{\partial u}(0,0) = 0$$

2.2

The first condition shows that u=0 is an equilibrium of equation (1) at $\mu = 0$. we are interested in local bifurcations that occur in the neighbourhood of this equilibrium when we vary the parameter μ. The second condition is a necessary, but not sufficient, condition for the appearance of local bifurcation at $\mu = 0$

Remarks 1.1

Suppose that the second condition is not satisfied: $\frac{\partial f}{\partial u}(0,0) \neq 0$

A direct application of the implicit function theorem shows that the equation $f(u,\mu) = 0$ posses a unique solution $u = u(\mu)$ in a neighbourhood of 0, for small enough μ. In particular u=0 is the only equilibrium of the equation (2.1) in a neighbourhood of zero when $\mu = 0$, and the same property hols for μ small enough. Furthermore, the dynamics of (2.1) in a neighbourhood of 0 is qualitatively the same for all sufficiently small values of the parameter μ: no bifurcation occurs for small values of μ.
Saddle-Node bifurcation
Theorem 1.1 (saddle-node bifurcation). Assume that the vector field f is of class $C^k, k \geq 2$, in a neighbourhood of (0,0) and satisfies:

$$\frac{\partial f}{\partial \mu}(0,0) =: a \neq 0, \frac{\partial^2 f}{\partial u^2}(0,0) =: 2b \neq 0$$

2.3

The following properties hold in the neighbourhood of 0 in $\mathbb{R}$ for small enough μ:

i. If $ab < 0$ (resptively $ab > 0$) the differential equation has no equilibria for $\mu < 0$ (resptively $\mu > 0$),

ii. If $ab < 0$ (resptively $ab > 0$) the differential equation posses two equilibria $\mu \pm (\epsilon), \epsilon = \sqrt{|\mu|}$ for > 0 (resptively $\mu < 0$), with opposite stabilities. Furthermore, the map $\epsilon \to u \pm (\epsilon)$ is of class C^{k-2} in a neighbourhood of 0 in $\mathbb{R}$, and $u \pm (\epsilon) = O(\epsilon)$. then for equation (2.1), a saddle-node bifurcation occurs at $\mu = 0$.

A direct consequence of conditions (2.3) is that f has the expansion:

$$f(u,\mu) = a\mu + bu^2 + o(|\mu| + u^2) \text{ as } (u,\mu) \to (0,0)$$

2.4

Pitchfork bifurcation

Theorem 2.2: consider a vector field f of class $C^k, K \geq 3$, in a neighbourhood of (0,0) that satisfies the condition (2.2), and that it is odd with respect to u

$$f(-u,\mu) = -f(u,\mu)$$

2.5

Furthermore assume that

$$\frac{\partial^2 f}{\partial \mu \partial u}(0,0) =: a \neq 0, \frac{\partial^3 f}{\partial u^3}(0,0) =: 6b \neq 0$$

2.6

i If $ab < 0$ (resptively $ab > 0$) the differential equation has one equilibrium for $u = 0$ for $\mu < 0$ (resptively $\mu > 0$). this equilibrium is stable when $b < 0$ and unstable when $b > 0$.

Ii. If $ab < 0$ (resptively $ab > 0$) the differential equation possesses the trivial equilibrium $u = 0$ and two non trivial equilibria $u \pm (\epsilon), \epsilon = \sqrt{|\mu|}$ for $\mu > 0$ (respectively $\mu < 0$),which are symmetric, $u_+(\epsilon) = -u_-(\epsilon)$. The map $\epsilon \to u_\pm(\epsilon)$ is of class C^{k-3} *in a neighbouhood* 0 *in* $\mathbb{R}$, and $u \pm (\epsilon) = O(t)$. The non trivial equilibria are stable when $b < 0$ and unstable when $b > 0$where as the trivial equilibrium has opposite stability.
Then for equation (2.1), a pitchfork bifurcation occurs at $\mu = 0$.
A direct consequence (2.2), (2.5) and (2.6) is that f has the Taylor expansion:
$f(u,\mu) = uh(u^2,\mu)h(u^2,\mu) = a\mu + bu^2 + 0(|\mu| + u^2)$ as $(u,\mu) \to (0,0)$ where h is of class $C^{(k-1)/2}$ in enough a neighbourhood of (0,0).

<u>Transcritical bifurcation</u>

Theorem 2.3: consider the vector field f of class $C^K, k \geq 2$, in a neighbourhood of (0,0) and that it satisfies condition 2.2, and also:

$\frac{\partial^2 f}{\partial\mu\partial u}(0,0) =: a \neq 0, \frac{\partial^2 f}{\partial u^2}(0,0) =: 2b \neq 0$

2.7

The following properties hold in the neighbourhood of 0 in $\mathbb{R}$ for small enough μ

i. The differential equation possesses the trivial equilibrium u=0 and the non trivial equilibrium $u_0(\mu)$ where the map $\mu \to u_0(\mu)$ is of the class C^{k-2} in a neighbourhood of 0 in $\mathbb{R}$, and $u_0(\mu) = O(\mu)$.

ii. If $a\mu < 0$ (resptively $a\mu > 0$) the trivial equilibrium u=0 is stable (respectively unstable) whereas the nontrivial equilibrium $u_0(\mu)$ is unstable(respectively stable).

Then for equation (2.1), a transcritical bifurcation occurs at $\mu = 0$. A direct consequence of condition (2.2) and (2.7) is that f has the Taylor expansion:

$f(u,\mu) = a\mu u + bu^2 + 0(u|\mu| + u^2)$ as $(u,\mu) \to (0,0)$

<u>Bifurcation of Dimension 2: Hopf bifurcation</u>

Here we consider Differential Equation in $\mathbb{R}^2$,

$\frac{du}{dt} = F(u,\mu)$

2.8

Here the unknown u is given a real-valued function that takes values in $\mathbb{R}^2$, and the vector field F is real-valued depending, besides u, upon a parameter μ. The bifurcation parameter. We assume that the vector field is of class $C^k, k \geq 3$, in a neighbourhood of (0,0) satisfying:

$F(0,0) = 0$

2.9

This condition ensures that u=0 is an equilibrium of equation (2.1) at $\mu = 0$. The occurrence of a bifurcation is in this case determined by linearization of the vector field at (0,0):

$$L = D_u F(0,0)$$

Which is a linear operator acting in $\mathbb{R}^2$. When L has eigenvalues on the imaginary axes, bifurcation may occur at $\mu = 0$. We focus in this section on the case where L has a pair of complex conjugated purely imaginary eigenvalues. This is called the Hopf bifurcation (or Andronov-Hopf bifurcation)

<u>Hypothesis</u> 2.1:

Assume that the vector field is of class $C^k, k \geq 5$, in a neighbourhood of (0,0), that is satisfies (2.9) and the two eigenvalues of the linear operator L are $\pm i\omega$ for some $\omega > 0$.

We consider the eigenvector and associated to the eigenvalue iω of L,

$L\xi = i\omega\xi$

If L^* is the adjoint operator of L then we define ξ^* as the eigenvector of L^* satisfying:

$L^*\xi^* = i\omega\xi^*, \quad \langle \xi, \xi^* \rangle = 1$

Where $\langle \cdot,\cdot \rangle$ denotes the Hermitian scalar product in $\mathbb{C}^2$. consider the Taylor extension of the vector field F in (2.8):

$F(U,\mu) =: \sum_{1 \leq r+q \leq} \mu^q F_{rq}(U^{(r)}) + o(|\mu| + \|U\|^k)$

Where F_{rq} is the r-linear symmetric operator from $(\mathbb{R}^2)^r$ to $\mathbb{R}^2$

$F_{rq} = \frac{1}{r!q!} \frac{\partial^q}{\partial \mu^q} D_u^r F(0,0)$

We define the coefficients

$a = \langle F_{11}\xi + 2F_{20}(\xi, -L^{-1}F_{01}), \xi^* \rangle$

2.10

$$b = \langle 2F_{20}(\xi, (2i\omega - L)^{-1}F_{20}(\xi,\xi) + 2F_{20}(\xi, -2L^{-1}F_{20}(\xi,\bar{\xi}) + 3F_{30}(\xi, \overset{\infty}{\tilde{\xi}}, \bar{\xi})), \xi^{*}\rangle \quad 2.11$$

Hypothesis 2.2

We assume that the complex coefficients a and b have non zero real parts, $a_r \neq 0$ *and* $b_r = 0$. The coefficient $b_r = R_r(b)$ is called the Lyapunov coefficient.

Definition 1.2

1. A non constant solution to the differential equation (2.8) is periodic if it exist T> 0 such that U(t) = U(t + T). The image of the interval [0,T] under U in the state space $\mathbb{R}^2$ is called the periodic orbit.
2. A periodic orbit Γ on a plane is called a limit cycle if it is theα − limit set of ω −limit set of some point z not on the periodic orbit, that is, the set of accumulation points of either forward or backward trajectory through z, is exactly Γ. Asymptotically stable and unstable periodic orbits are examples of limit cycles.
3.

Theorem 2.1 (Hopf Bifurcation)

Assume that hypothesis 2.1 and 2.2 holds. Then, for the differential equation(2.1) a Superciritcal (respectively Subcritical) Hopf Bifurcation occurs at μ = 0 when $b_r < 0$ (*respectively* $b_r > 0$). More precisely, the following properties hold in a neighbourhood of O in $\mathbb{R}^2$ for small enough μ:

i. If $a_r b_r < 0$ (*respectively* $a_r b_r > 0$) the differential equation has precisely one equilibrium u(μ) for μ < 0 (respectively μ > 0) with μ > 0 with u(0)=0. This equilibrium is stable when $b_r < 0$ and unstable when $b_r > 0$.

ii. If $a_r b_r < 0$ (*respectively* $a_r b_r > 0$) the differential equation possesses for u(μ) for μ < 0 (respectively μ > 0) and equilibrium u(μ) and a unique periodic orbit $U^{*}(\mu) = O(\sqrt{|U|})$, which surrounds this equilibrium. The periodic orbit is stable when $b_r < 0$ and unstable when $b_r > 0$, whereas the equilibrium has the opposite stability.

Remark 2.2

The number of equilibria of the differential equation stays constant upon varying μ in neighbourhood of O. The dynamics of the bifurcation change at the bifurcation point μ = 0. Such bifurcation, are called dynamic bifurcations, whereas those in which the number of equilibria changes are also called steady bifurcation.

Hopf bifurcation theorem for vector fields

Let X_μ be a C^k $(k \geq 4)$ vector field on $\mathbb{R}^2$ such that $X_\mu(0) = 0$ for all μ and X=$(X_\mu, 0)$ is also C^k. Let $dX_\mu(0,0)$ have two distinct, simple complex conjugate eigenvalues λ(μ) and $\overline{\lambda(\mu)}$ such that μ < 0, Re λ(μ) < 0 , for μ = 0, Re λ(μ) = 0, and for μ > 0, Re λ(μ) > 0. Also assume $\left.\frac{d\mathrm{Re}\,\lambda(\mu)}{du}\right|_{\mu=0} > 0$. Then there is a C^{k-2} function μ: $(-\epsilon, \epsilon) \to \mathbb{R}$ such that $(X_1, 0, \mu(X_1))$ is on a closed orbit of period $\approx \frac{2\pi}{|\lambda(0)|}$ and radius growing like $\sqrt{\mu}$, of the flow of X for $X_1 \neq 0$ and such that μ(0) = 0. There is a neighbourhood u of (0,0,0) in $\mathbb{R}^3$ such that any closed orbit in u is one of the above.

Furthermore, if 0 is a 'vague attractor' (asymptotically stable) for X_o, *then* $\mu(X_1) > 0$ for all $X_1 = 0$ and the orbit is attracting.

If, instead of a pair of conjugate eigenvalues crossing the imaginary axis, a real eigenvalue crosses the imaginary axis, two stable fixed point will branch off instead of a closed orbit.

Center manifold theorem

The center manifold theorem is one of the important bifurcation theorem and the key job is that it enables one to reduce to a finite dimensional problem. In the case of a Hopf bifurcation theorem, it enables a reduction to two dimension without loosing any information concerning stability.

Theorem 2.2 (Center manifold theorem): Let ψ be a mapping of a neighbourhood of zero in a Banach space Z into Z. We assume that ψ is C^{k+1}, k≥ 1 and that $\psi(0) = 0$. we further assume that Dψ(0) has spectral radius 1 and that the spectrum of Dψ(0) splits into a part on the unit circle and the remainder which is at a non-zero distance form the unit circle. Let Y denote the generalized eigenspace of Dψ(0) belonging to the part of the spectrum on the unit circle; assume that Y has dimension d< ∞ then there exist a neighbourhood v of O in Z and a C^k submanifold M of v of dimension d, passing through O and tangent to Y at O. such that

a. **Local invariance:** If x∈ M and ψ(x) ∈ V then ψ(x) ∈ M

b. **Local attractively:** If $\psi^n(x) \in V$ for every n=0,1,2,…, then as n→ ∞, the distance from $\psi^n(x)$ to M→ 0. this holds automatically if Z is finite dimensional or, more generally, if Dψ(0) is compact.

Existence and Uniqueness

1. Lipschitz conditions

Consider $\frac{dy}{dt} = f(t,y)$

Y(t_0)=y_0

Where f is a differentiable function. We would like to know when we have existence of a unique solution for given initial date. One condition on f which guarantees this in the following

Given a subset S of the (t,y)-plane, we say that f is lipschitz with respect to y on the domain s if there exist some constant k such that

$|f(t,y_2) - f(t,y_1)| \leq k|y_2 - y_1|$ for every point (t,y_1) and (t,y_2) in S. The constant K is called the Lypschitz constant.

Example 2.1 Let f(t,y)=ty^2

then since $|f(t,y_2) - f(t,y_1)| \leq f|y_2 + y_1||y_2 - y_1|$ is not bounded by any constant times $|y_2 - y_1|$, f is not Lipschitz continuous with respect to y on the domain ℝXℝ. However f is lipschitz on any rectangle $\mathbb{R} = [a,b]x[cxd]$ since we have

$t|y_2 + y_1| \leq 2\max\{|a|,|b|\}.\max\{|c|,|d|\}$ on ℝ.

1. $d(x,y) \geq 0$
2. $d(x,y) = 0$ iff $x = y$
3. $d(x,y) = d(y,x)$
4. $d(x,z) = d(x,y) + d(y,$

Floquet theory:

The fundamental matrix x(t) of

$\dot{x} = \dot{A}(t)x$ 2.12

With $x(t) = 1$, has a Floquet normal form

$x(t) = Q(t)e^{Bt}$ 2.13

where $Q \in C^1(\mathbb{R})$ is T periodic and the matrix B, $B \in \mathbb{C}^{nxn}$ satisfies the equation

$C = X(T) = e^{BT}$ 2.14

$Q(0) = 1$ and $Q(t)$ is an invertible matrix for all t.

PROOF:

By **lemma 2.5**, there exist a non singular constant matrix C with x(t+T)= X(t)C using X(t+T) = X(t)X(T) = x(t)C and Lemma 2.6 gives $C=X(T)=-e^{BT}$

For some matrix B, if $Q(t) = X(t)e^{BT}$, then for all t,

$$\begin{aligned} Q(t+T) &= x(t+T)e^{-B(t+T)} \\ &= x(t)Ce^{-Bt}e^{-BT} \\ &= x(t)e^{BT}e^{-Bt}e^{-BT} \\ &= x(t)e^{-Bt} \\ &= Q(t) \end{aligned}$$

This means that

$x(t) = Q(t)e^{Bt}$ where

$Q \in C^1(\mathbb{R})$ is T-periodic and $x(0) = X(0)e^0 = 1$

The matrix e^{-Bt} is invertible for all t, because exponential of square matrices are invertible and x(t) is invertible Hence, Q(t) is invertible.

Lemma 2.3

If x(t) is a fundamental matrix of (1), then so is Y(t) = x(t)B for non singular constant matrix B.

Lemma 2.4

If x(t) is a fundamental matrix of (1), then so is Y(t) = X(t + T)

Lemma 2.5

If x(t) is a fundamental matrix of $\dot{x} = A(t)x$ *by lemma* 2.4, Y(t) = X(t + T) is a fundamental matrix of (1), then there exist a non singular constant matrix C with

$X(t+T) = X(t)C$ (2.15)

Floquet Multiplier

We continue using the fundamental matrix X(t) for (1) in Lemma 2.5, we proved that $X(t+T) = X(t)C$

Where C is a non singular constant matrix. Recall in (5)

$C = C(0) = x^{-1}(0)Y(0) = x^{-1}(0)x(T)$ (2.16)

This C is known as the monodromy matrix.

Definition 2.3

The eigenvalues of the monodromy matrix are called the Floquet multiplier of (1)

Definition 2.4

The eigenvalues of the matrix B of the Floquet form $x(t) = Q(t)e^{Bt}$, are called the Floquet exponents of (1). Since the monodromy matrix is non singular, its eigenvalues are non zero, therefore, we can state the following:

Corollary 2.5

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the Floquet multipliers and $\mu_1, \mu_2, \ldots, \mu_n$ be the Floquet exponents for (1), we can write

$\lambda_j = e^{\mu_j t}$ for all j= 1,…,n

Stability of the Floquet System

Floquet multipliers are very useful in stability analysis of periodic system. Recall the following definitions.

Definition 3.5:

An eigenvalue λ of A is simple if its algebraic multiplicity equals 1.

Definition 3.6:

Let λ be an eigenvalue of a matrix A the geometric multiplicity of λ is dim(Null (A-λI)) in order words, the number of linearly independent eigenvector associated with λ.

Definition 3.7:

An eigenvalue λ of A is semi simple if its geometric multiplicity equals its algebraic. A simple eigenvalue is always semi-simple. But the converse is not true.

Definition 3.8

Consider the system $\dot{x} = A(t)x \ \ in\ v = [t_0, \infty]$ (2.17)

and assume A(t) is T periodic and continuous in V. The solution ψ(t) to system (2.12) is

1. Stable on v if for every $\epsilon > 0$, there exist a $\delta > 0$, such that $|\Psi(t_0) - x(t_0)| < \delta$ implies that $|\Psi(t) - x(t)| < \epsilon, for\ every$ t≥ 0

And the solution x(t) is defined for all t ∈ v

2. Asymptotically stable on v If it is stable and if in addition $\lim_{t\to\infty}|\psi(t) - x(t)| \longrightarrow 0$
3. Unstable if it is not stable on v. it can be proven that the following stability condition hold for the Floquet system.

Theorem 2.10

Assume $\lambda_1, \lambda_2, \ldots, \lambda_n$ are Floquet multipliers of system

1. Then the zero solution of (1) is
   i. Asymptotically stable on [0,∞) if and only if $|\lambda_i| < 1$ for i=1,...,n
   Ii. Stable on $[0, \infty)$ if $|\lambda_j| \leq 1$ for all I=1,...,n and whenever $|\lambda_j| = 1, \lambda_j$ is a semi-simple eigen value
   iii. Unstable in all other cases

It should be noted that for the Floquet exponents, the conditions $|\lambda_j| < 1, |\lambda_j| \leq 1, |\lambda_j| > 1$ is equivalent to Re $\mu_j < 0$, Re $\mu_j \leq 0$, and Re $\mu_j > 0$

Eigenvalue

Eigenvalue are a special set of scalars associated with a linear system of equation (ie a matrix equation) that are sometimes known as characteristic roots, characteristic value (Hofman &Kunac 1971), proper value, or latent roots (Marcus & Minc 1988, p.144)

Theorem 2.1

The following gives the link between the characteristic polynomial of a matrix A and its eigenvalues. If A is an nxn matrix and λ is a complex number the the following are equivalent

a. λ is an eigenvalue of A
b. The system of equation (A-λI=0) has a trivial solution
c. There is a non zero vector X in $\mathbb{C}^n$ such that Ax=λx
d. λ is a solution of the characteristic equation $\det(A - \lambda I)$. Some coefficient of the characteristic polynomial of A have a specific shape. The following theorem gives the information about it.

Theorem 2.1.2

If A is an n x n matrix, then the characteristic polynomial P(λ) of A has degree n, the coefficient of $\lambda^n\ is\ (-1)^n$, the coefficient of λ^{n+1} is $(-1)^{n-1}$ trace (A) and the constant term is det(a), where trace (A)=$a_{11} + a_{22} + \ldots + a_{nn}$. In some structured matrices, eigenvalues can be read as shown in theorem 2.1.3.

Theorem 2.1.3

If a is an nxn triangular matrix (upper triangular, lower triangular, or diagonal), then the eigenvalues of A are entries of the main diagonal of A.

Cayley-Hamilton's theorem is one of the most important statements in linear algebra. The Theorem states that

Theorem 2.1.4

Substituting the matrix A for λ in characteristic polynomial of A, we get the result of zero matrix ie, P(A)=0

Jacobian Theorem

If u and v are functions of the two independent variables of x and y,

the determinant $\begin{pmatrix} \frac{\partial u \partial u}{\partial x \partial y} \\ \frac{\partial v \partial v}{\partial x \partial y} \end{pmatrix}$ is called the Jacobian of u,v with respect to x,y and is written as $\frac{\partial(u,v)}{\partial(x,y)}$ $or\ J(\frac{u,v}{x,y})$

Properties of Jacobian

First property

If U and V are the function of x and y then $\frac{\partial(u,v)}{\partial(x,y)}$X $\frac{\partial(x,y)}{\partial(u,v)} = 1$

Second Property

If U,V are the functions of r,s where r and s are function of x,y, the $\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(r,s)}$X $\frac{\partial(r,s)}{\partial(x,y)}$

Third Property

If function U,V,W of three independent variables x,y,z are not independent then

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = 0$$

Recently, Rachunkova in [32] and Torres in [8] studied periodic boundary problems by using signed Green's function combining Kransnoselskii's fixed point theorem on compression and expansion of cones, and they obtained the new existence and multiplicity result concerning one signed periodic solution of the equation as well as equations with singularity. But the method mentioned above is difficult to be applied for estimate of the sharp number of solutions of (1.1), because it is impossible to determine the sharp norm for the Green's function. For the case indefinite weight, even the existence of T- periodic solution is not known. It seems that infinite dimensional singularity theory established by Berger and Church in [4] provides a natural platform to deal with such kind of problems, and has been already successfully applied to non homogeneous non linear elliptic equations with both Dirchlet and Neumann boundary values respectively. To know more about this approach, one can refer to the approach, one can refer to comprehensive survey articles [10,33].

In order to understand the global structure of periodic solutions and stability of each solutions under the cubic restoring force, Chen and Li in [2] used the different approach based on Crandall-Robinowitz bifurcation theorem and contraction method but devoted their work to the exact multiplicity and stability of periodic solution under cubic nonlinear restoring force with a strong damped condition.

$a(t) \leq \frac{(\pi)^2}{T^2} + \frac{C^2}{4}$, and $\bar{a} > 0$ where $\bar{a}$ denotes the average of a(t) over a period.
In this paper they used global bifurcation method to cover the situation that the method to cover the situation that the method based on maximum principle developed in [37, 30, 29] and super-sub solutions methods are not applicable to degenerate situation and the interesting case when the graph of forcing term h(t) intersect the critical level is missing where bifurcation occurs. There result confirmed the first issue of the number of periodic solution of (1.1) is at most three under strong damped condition. Generally exactly one or three. The periodic solution of (1.1) forms an "S" - shape smooth curve, symmetric with respect to the origin.
Dutta and Prajapatic in [3] reported a symmetric investigation in the phase space of the double well Duffing Oscillator, they used bifurcation diagram to show the region characterized by the parameters for which one finds periodic solutions, a periodic solution. They also observed that when driving force is increased, there is a series of parallel "islands" of parameters characterized by a periodic attractors. They found that even the model is perturbed by linear term, it shows periodic and chaotic behaviour and that when damping coefficient is taken as zero and the non linear stiffness parameter is taken sufficiently small, the model shows homoclinic nature for whatever the value of force.

4. Results and Discussion

Our modified Duffing equation is

$$\ddot{x} + \alpha\dot{x} - \tau^2 x + \beta x^3 = f\sin\omega t \quad (4.1)$$

With initial conditions $x(0) = 1$ and $\dot{x}(0) = 1$
The equivalent of (4.1) is given by

$$\dot{x} = y,$$

$$\dot{y} = -\propto y - \tau^2 x + \beta x^3 + F\sin wt \quad (4.2)$$

Let $X_\alpha(x, y) = (y, -\propto y - \tau^2 x + \beta x^3 + F\sin wt)$
Now $X_\alpha(0,0) = 0$ for every α and

$$dx_\alpha(0,0) = \begin{pmatrix} 0 & 1 \\ -\tau^2 & -\alpha \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ -\tau^2 & -\alpha \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 0$$

$$\begin{pmatrix} 0 & 1 \\ -\tau^2 & -\alpha \end{pmatrix} - \lambda\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = 0$$

$$\begin{pmatrix} -\lambda & 1 \\ -\tau^2 & -\alpha-\lambda \end{pmatrix} = 0 \Rightarrow (-\lambda)(-\alpha-\lambda) - -\tau^2 = 0$$

$$\alpha\lambda + \lambda^2 + \tau^2 = 0$$

$$\lambda^2 + \alpha\lambda + \tau^2 = 0 \quad (4.3)$$

$$\lambda = \frac{-\alpha \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\lambda(\alpha) = \frac{-\alpha \pm \sqrt{\alpha^2 - 4\tau^2}}{2} \quad (4.4)$$

Consider $\propto$ such that $|\propto| < 2$
In this case $\frac{-\propto}{2} + \frac{i\sqrt{\propto^2 - 4\tau^2}}{2}$
$im\lambda(\alpha) \neq 0$,
Where $\lambda(\alpha) = \frac{-\alpha \pm \sqrt{\alpha^2 - 4\tau^2}}{2} = \frac{-\propto}{2} + \frac{i\sqrt{\propto^2 - 4\tau^2}}{2}$
Furthermore, for $-2 < \propto < 0$, $\text{Re}\,\lambda(\alpha) < 0$
and for $\propto = 0$, $\text{Re}\,\lambda(\alpha) = 0$ and for $2 > \propto > 0$,
$\text{Re}\,\lambda(\alpha) > 0$ and $\left.\frac{d(R_e\lambda(\alpha))}{d\propto}\right|_{\propto=0} = -\frac{1}{2}$

Therefore the Hopf bifurcation theorem applies and we conclude that there is one parameter family of closed orbits of x=$(x_\alpha, 0)$ in a neighbourhood of (0,0,0)
To find out if these orbits are stable and if they occur for $\alpha > 0$, we look at
$X_0(x,y)=(y, -\tau^2 x + \beta x^3 + Fsinwt)$.
$dx_0(0,0) = \begin{pmatrix} 0 & 1 \\ -\tau^2 & 0 \end{pmatrix}$ and $\lambda(0) = \tau i$
Recall that to use the stability formular in we must choose coordinate so that
$dx_\alpha(0,0) = \begin{pmatrix} 0 & Im(\lambda_0) \\ -Im(\lambda_0) & -\alpha \end{pmatrix}=\begin{pmatrix} 0 & \tau \\ -\tau & 0 \end{pmatrix}$
Which is not in the required form. We must make a change of coordinates so that
$dx_0(0,0)$ becomes $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ that is we must find vectors $\hat{e_1}$ and $\hat{e_2}$ so that $dx_0(0,0)\hat{e_1} = -\hat{e_2}$ and $dx_0(0,0)\hat{e_2} = \hat{e_1}$
The vectors $\hat{e_1} = (1, -1)$ and $\hat{e_2} = (0,1)$ will do.
A procedure for finding $\hat{e_1}$ and $\hat{e_2}$ is to find $\propto$ and $\bar{\propto}$ the complex eigenvectors. We may then take $\hat{e_1} = \propto + \bar{\propto}$ and $\hat{e_2} = i(\propto - \bar{\propto})$

$$X_0(x\hat{e_1} + y\hat{e_1}) = X_0(x, y - x)$$
$$= (y, -\propto (y - x) - \tau^2 x + \beta x^3 + F\sin\omega t)$$
$$= (y\hat{e_1}, -\propto (y - x) + \tau^2 x + \beta x^3 + F\sin\omega t)\hat{e_2} \quad (4.6)$$
$$\therefore\therefore\therefore X_0(x, y) = (y, -\propto (y - x) + \tau^2 x + \beta x^3 + F\sin\omega t)$$

$\frac{\partial^n x_1}{\partial x^j \partial y^{n-j}}$ (0,0)=0 for every n> 1

$$\therefore x_1(x, y) = y \quad (4.7)$$
$$X_2(x, y) = -\propto (y - x) + \tau^2 x + \beta x^3 + F\sin\omega t \quad (4.8)$$

$\therefore \frac{\partial^2 x_2}{\partial y^2}(0,0)=0$, $\frac{\partial^2 x_2}{\partial x \partial y}(0,0)=0$
$\frac{\partial^3 x_2}{\partial x^3}(0,0)=6\beta$, $\frac{\partial^3 x_2}{\partial x^2 \partial y}(0,0)=0$
$\frac{\partial^3 x_2}{\partial x \partial y^2}(0,0)=0$, $\frac{\partial^3 x_2}{\partial y^3}(0,0)=0$

$$\therefore \ddot{v}(0) = \frac{3\pi}{4|\lambda(0)|}(6\beta) \quad (4.9)$$

The orbits are unstable and bifurcation takes place below criticality. The orbits occur for $\mu < \mu_0$ and are repelling on the center manifold, and so are unstable by general.
We illustrate the numerical simulation of the results using the MATHCAD software:

5. Conclusion

From our results, the Floquet method is very effective in determining the stability and Hopf bifurcation analysis of the periodic solution of the Duffing oscillator. The advantage of this method is that it shows the regions where orbits are stable and unstable. The bifurcation points showed critical and sub-critical regions. The orbits were found to be unstable and are repelling each other at the center of the manifold.

Conflict of interest

The authors declare no conflict of interest regarding the publication of their paper

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