Innovations

Study of $\Re\mathcal{M}$ Distribution Generalization and Certain Common Theorems

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Abstract: *In this paper, we study a significant number of distributional theoremsby considering linear ordinary differential equations and we use RAHMOH(will call it $\Re\mathcal{M}$ transform) transform, we find that $\Re\mathcal{M}$ integral transform is convenient and crucial to use to get an essential result.*

1 Introduction:

The need for a theory of distributions arose from the inadequacy of certain methods of classical analysis regard to some applications. Thus a number of procedures which were being employed with success by physicists could not be justified rigorously within the framework of existing theories. The most striking example, and one that we shall use as the guideline in our construction of distributions, is the differentiation of non-continuous functions—a famous example being the Dirac delta function which is the "derivative" of the Heaviside function.

The $\Re\mathcal{M}$ transform defining on the set

$$A = \left\{ f(t) : \exists M, k_1, k_2 > 0, |f(t)| < Me^{\frac{|t|}{q_i}} \ ,\ \text{if } t\epsilon(-1)^i \times [0,\infty) \right\}$$

Which M is constant must be finite number, k_1, k_2 may be finite or infinite. by the following integral:

$$\mathcal{M}[f(t)](s,u) = F(s,u) = u^2 \int_0^\infty f(t) e^{\frac{-st}{u}} dt \quad ; \quad s > 0, u > 0$$

2 Preliminaries:

In this section, we will introduce the basic knowledge and concepts, which are essential for this work.

2.1 Definition:

Let $\mathcal{D}$ be the space consisting of all real-valued functions $\varphi(t)$ with continuous derivatives of all orders and Compact support. The support of $\varphi(t)$ is the closure of the set of all elements $t \in R$ such that $\varphi(t) \neq 0$. Then $\varphi(t)$ is called a test function.

2.2 Definition:

A distribution T is a continuous linear functional on the space $\mathcal{D}$. The space of all such distributions is denoted by $\mathcal{D}'$.

For every $T \in \mathcal{D}'$and $\varphi(t) \in \mathcal{D}$, the value that T acts on $\varphi(t)$ is denoted by $\langle T, \varphi(t) \rangle$. Note that $\langle T, \varphi(t) \rangle \in R$.

2.3 Definition:(Locally Integrable function for Test Function)

Let Ω be an open set in the Euclidean space R^n. Then a function $f: \Omega \longrightarrow \mathbb{C}$such that $\int_\Omega |f\varphi|\, dx < +\infty$, for each test function $\varphi \in \mathbb{C}_c^\infty(\Omega)$ is called locally Integrable , and the set of such functions is denoted by $L_{1,loc}(\Omega)$.

Here $\mathbb{C}_c^\infty(\Omega)$ denotes the set of all infinitely differentiable functions $\varphi: \Omega \longrightarrow R$with compact support contained inΩ.

Examples of locally integrable functions:

(i) The function 1 defined on the real line is locally integrable function. More generally, constants, continuous functions and integrable functions are locally integrable.

(ii) The function $f(x) = \frac{1}{x}$, $x \in (0,1)$ is locally integrable.

Examples of Distributions:

(i) The locally integrable function $f(t)$ is a distribution generated by the locally integrable function$f(t)$.

Then we define $\langle T, \varphi(t) \rangle = \int_\Omega f(t)\varphi(t)\, dt$ where Ω is support of $\varphi(t)$ and $\varphi(t) \in \mathcal{D}$.

(ii) The Dirac delta function id distribution defined by $\langle \delta(t), \varphi(t) \rangle = \varphi(0)$, and the support of $\delta(t)$is $\{0\}$.

A distribution T generated by a locally integrable function is called a regular distribution, otherwise, it is called a singular distribution.

2.4 Definition:

The K^{th}-order derivative of distribution T denoted by $T^{(k)}$, is defined by:

$$\langle T^{(k)}, \varphi(t) \rangle = (-1)^k \langle T, \varphi^{(k)}(t) \rangle \quad \forall \varphi(t) \in \mathcal{D}$$

Example:

(i) $\langle \delta'(t), \varphi(t) \rangle = -\langle \delta(t), \varphi'(t) \rangle = -\varphi'(0)$

(ii) $\langle \delta^{(k)}(t), \varphi(t) \rangle = (-1)^k \langle \delta(t), \varphi^{(k)}(t) \rangle = (-1)^k \varphi^{(k)}(0)$

2.5 Definition:

Let $\alpha(t)$ be an infinitely differentiable function, we define the product of $\alpha(t)$ with any distribution T in $\mathcal{D}'$ by:

$$\langle \alpha(t)T, \varphi(t)\rangle = \langle T, \alpha(t)\varphi(t)\rangle \ , \qquad \forall \varphi(t) \in \mathcal{D}$$

2.6 Definition:

Let $\lambda \in R$ and $f(t)$ be a locally integrable function satisfying the following conditions:

(i) $f(t) = 0$ for all $t < \lambda$

(ii) There exist a real number $M > 0$ such that $|f(t)| < Me^{\frac{|t|}{q_i}}$

(iii) There exist a real number c, h such that $e^{\frac{-ct}{h}} f(t)$ is absolutely integrable over R.

The $\Re\mathcal{M}$ transform of $f(t)$ is defined by:

$$F(s,u) = \mathcal{M}[f(t)] = u^2 \int_{\lambda}^{\infty} f(t)\, e^{\frac{-st}{u}} dt$$

Where s, u are complex variables.

It is known that if $f(t)$ is continuous,then $F(s,u)$ is an analytic function on the half plane $R(s,u) > \sigma_a$, where σ_a is an abscissa of absolute converges of $\mathcal{M}[f(t)]$.

Recall the $\Re\mathcal{M}$ transform $G(s,u)$ is defined by

$$G(s,u) = \mathcal{M}[g(t)] = u^2 \int_{\lambda}^{\infty} g(t)\, e^{\frac{-st}{u}} dt$$

Where $\mathfrak{R}(s,u) > \sigma_a$, can be written as $\langle u^2 g(t), e^{\frac{-st}{u}} \rangle$.

2.7 Definition:

Let $\mathbb{S}$ be the space of test functions of rapid decay containing the complex-valued function $\varphi(t)$ having the following properties:

(i) $\varphi(t)$ is infinitely differentiable, i.e. $\varphi(t) \in \mathbb{C}^\infty(R)$.

(ii) $\varphi(t)$, as well as its derivatives of all orders, vanish at infinitely faster than the reciprocal of any polynomial which is expressed by inequality: $\left|t^p \varphi^{(k)}(t)\right| < C_{pk}$

Where C_{pk} is constant depending on p, k and $\varphi(t)$,then $\varphi(t)$ is called a test function in the space $\mathbb{S}$.

2.8 Definition:

A distribution of slow growth or tempered distribution T, is a continuous linear function over the space $\mathbb{S}$ of test functions of rapid decay and contained the complex-valued functions, i.e. there is assigned a complex number $\langle T, \varphi(t)\rangle$ with properties:

(i) $\langle T, c_1\varphi_1(t) + c_2\varphi_2(t)\rangle = c_1\langle T, \varphi_1(t)\rangle + c_2\langle T, \varphi_2(t)\rangle$ for $\varphi_1(t), \varphi_2(t) \in \mathbb{S}$ and c_1, c_2 are constants.

(ii) $\lim_{m\to\infty} \langle T, \varphi_m(t)\rangle = 0$ for every sequence $\{\varphi_m(t)\} \in \mathbb{S}$.

We shall let $\mathbb{S}'$ denote the distribution of slow growth.

3 Main Results:

3.1 Definition:

Let $f(t)$ be a distribution satisfying the following properties:

(i) $f(t)$ is a right-sided distribution, that is, $f(t) \in \mathcal{D}'_R$

(ii) There exist a real number$c\,,h$ such that $e^{\frac{-ct}{h}} f(t)$ is tempered distribution.

The $\Re\mathcal{M}$ transform of right-sided distribution $f(t)$ satisfying (b) is defined by:

$$F(s,u) = \mathcal{M}[f(t)] = \langle u^2 e^{\frac{-ct}{u}} f(t), X(t) e^{\frac{-(s-c\}t}{u}}\rangle \quad (*)$$

Where $X(t)$ is an infinitely differentiable function with support bounded on the left, which equals 1 over a neighborhood of the support of $f(t)$.

For $R(s,u) > c$, the function $X(t)e^{\frac{-(s-c\}t}{u}}$ is a testing function on the space $\mathbb{S}$, and $e^{\frac{-ct}{u}} f(t)$ is in the space $\mathbb{S}'$.

Equation (*) can be reduced to:$F(s,u) = \mathcal{M}[f(t)] = \langle u^2 f(t), e^{\frac{-st}{u}}\rangle$

Now $F(s,u)$ is a function of $s\,,u$ defined over the right half-plane $R(s,u) > c$.

All these results have been proved in the references books.

3.2 Theorem:

Let $\delta(t)$ be the Dirac delta function, $H(t)$be the Heaviside function,and $f(t)$ be a $\Re\mathcal{M}$ transformable distribution in $\mathcal{D}'_R$. If k is a positive integer, then the following holds:

(i) $\boldsymbol{\mathcal{M}\left(t^{k-1}\frac{H(t)}{(k-1)!}\right) = \frac{u^{k+2}}{s^k}}\,, \boldsymbol{\Re(s)} > 0$

Proof:

By using $\Re\mathcal{M}$ definition we find:

$$\mathcal{M}\left(t^{k-1}\frac{H(t)}{(k-1)!}\right) = u^2\int_0^\infty t^{k-1}\frac{H(t)}{(k-1)!}\, e^{\frac{-st}{u}}\, dt$$
$$= \frac{u^2}{(k-1)!}\int_0^\infty t^{k-1}H(t)\, e^{\frac{-st}{u}}\, dt, t > 0$$

And since $H(t) = \begin{cases} 1\,, & t \geq 0 \\ 0\,, & t < 0 \end{cases}$

Then $\mathcal{M}\left(t^{k-1}\frac{H(t)}{(k-1)!}\right) = \frac{u^2}{(k-1)!}\int_0^\infty t^{k-1}\, e^{\frac{-st}{u}}\, dt = \frac{u^{k+2}(k-1)!}{s^k(k-1)!} = \frac{u^{k+2}}{s^k}$.

(ii) $\boldsymbol{\mathcal{M}(\delta(t)) = u^2}\,, -\infty < R(\boldsymbol{s},\boldsymbol{u}) < \infty$

Proof:

$$\mathcal{M}(\delta(t)) = u^2\int_0^\infty \delta(t)\, e^{\frac{-st}{u}}\, dt = u^2\, e^{\frac{-s(0)}{u}}\int_0^\infty \delta(t)\, dt = u^2\cdot 1 = u^2.$$

(iii) $\boldsymbol{\mathcal{M}\left(\delta^{(k)}(t)\right) = \frac{s^k}{u^{k-2}}}\,, -\infty < R(\boldsymbol{s},\boldsymbol{u}) < \infty$

Proof:

$$\mathcal{M}\left(\delta^{(k)}(t)\right) = u^2\int_0^\infty \delta^{(k)}(t)\, e^{\frac{-st}{u}}\,dt = u^2\frac{d^k}{dt^k}\left(e^{\frac{-st}{u}}\right)\Big|_{t=0}\int_0^\infty \delta(t)\,dt$$
$$= \frac{s^k}{u^{k-2}}\cdot\, e^{\frac{-s(0)}{u}}\cdot 1 = \frac{s^k}{u^{k-2}}.$$

(iv) $\boldsymbol{\mathcal{M}\left(tf^{(k)}(t)\right) = \frac{u^2}{s}\frac{d}{du}\left[\mathcal{M}\left[f^{(k)}(t)\right]\right] - \frac{u^2}{s}\left[\mathcal{M}\left[f^{(k)}(t)\right]\right]}$

Proof:

$$\frac{d}{du}\left[\mathcal{M}\left[f^{(k)}(t)\right]\right] = \frac{d}{du}\int_0^\infty u^2 f^{(k)}(t)\, e^{\frac{-st}{u}}\,dt = \int_0^\infty \frac{\partial}{\partial u}\left[u^2 f^{(k)}(t)\, e^{\frac{-st}{u}}\right]dt$$
$$= 2u\int_0^\infty f^{(k)}(t)e^{\frac{-st}{u}}\,dt + s\int_0^\infty tf^{(k)}(t)e^{\frac{-st}{u}}\,dt$$
$$= \frac{2\mathcal{M}\left[f^{(k)}(t)\right]}{u} + \frac{s}{u^2}\mathcal{M}\left(tf^{(k)}(t)\right)$$

So,

$$\mathcal{M}\left(tf^{(k)}(t)\right) = \frac{u^2}{s}\frac{d}{du}\left[\mathcal{M}\left[f^{(k)}(t)\right]\right] - \frac{u^2}{s}\left[\mathcal{M}\left[f^{(k)}(t)\right]\right].$$

Theorems Regarding to $\mathfrak{R}\mathcal{M}$ transform:

3.3 Theorem: (Laplace-$\mathfrak{R}\mathcal{M}$ transform duality)

Let $\mathcal{L}[f(t)] = F(s)$ and $\mathcal{M}[f(t)] = M(s,u)$ be respectively Laplace and $\mathfrak{R}\mathcal{M}$ transforms of $f(t) \in A$. Then $M(s,u) = u^2F\left(\frac{s}{u}\right)$.

Proof: By using definition of $\mathfrak{R}\mathcal{M}$ transform:

$$\mathcal{M}[f(t)] = u^2\int_0^\infty e^{\frac{-st}{u}} f(t)dt = u^3\int_0^\infty e^{-st} f(ut)dt$$

Setting $\alpha = ut$, $t = \alpha/u$ and $dt = d\alpha/u$. Then

$$\frac{1}{u}\left\{u^3\int_0^\infty e^{\frac{-s\alpha}{u}} f(\alpha)d\alpha\right\} = u^2\int_0^\infty\int_0^\infty e^{\frac{-s\alpha}{u}} f(\alpha)d\alpha = u^2F\left(\frac{S}{u}\right)$$

3.4 Theorem:

Let the functions $f(t), g(t) \in A$. If $M(s,u)$ and $N(s,u)$ are respectively the $\mathfrak{R}\mathcal{M}$ transforms of $f(t)$ and $g(t)$. Then the convolution theorem of RAHMOH transform is given by:

$$\mathcal{M}[(f * g)(t)] = \frac{1}{u^2}M(s,u)N(s,u)$$

Where $f * g$ is the convolution of two functions $f(t)$ and $g(t)$ which is defined by: $\int_0^t f(\alpha)g(t-\alpha)d\alpha = \int_0^t f(t-\alpha)g(t)d\alpha$

Proof: Let firstly recall the Laplace transform of $(f * g)(t)$ defined as:

$$\mathcal{L}[(f * g)(t)] = F(s)G(s)$$

By using Theorem (2) above, we have $\mathcal{M}[(f * g)(\mathrm{t})] = \mathrm{u}^2\mathcal{L}[(\mathrm{f} * \mathrm{g})(\mathrm{t})]$, thus

$$\mathcal{M}[(\mathrm{f} * \mathrm{g})(\mathrm{t})] = \mathrm{u}^2\mathcal{L}[(\mathrm{f} * \mathrm{g})(\mathrm{t})] = \mathrm{u}^2\mathrm{F}\left(\frac{\mathrm{S}}{\mathrm{u}}\right)\mathrm{g}\left(\frac{\mathrm{S}}{\mathrm{u}}\right)$$
$$= \frac{1}{\mathrm{u}^2}\mathrm{u}^2\mathrm{F}\left(\frac{\mathrm{S}}{\mathrm{u}}\right)\mathrm{u}^2\mathrm{g}\left(\frac{\mathrm{S}}{\mathrm{u}}\right) = \frac{1}{\mathrm{u}^2}\mathrm{M}(\mathrm{s},\mathrm{u})\mathrm{N}(\mathrm{s},\mathrm{u})$$

3.5 Theorem: (Natural-$\Re\mathcal{M}$ transform duality)

Let $N[f(t)] = R(s,u)$ and $\mathcal{M}[f(t)] = M(s,u)$ be respectively Natural and $\Re\mathcal{M}$ transforms of $f(t) \in A$. Then $M(s,u) = u^3 R(s,u)$.

The proof is same as theorem 3.3.

3.6 Theorem: (Sumudu-$\Re\mathcal{M}$ transform duality)

Let $S[f(t)] = G(s,u)$ and $\mathcal{M}[f(t)] = M(s,u)$ be respectively Sumudu and $\Re\mathcal{M}$ transforms of $f(t) \in A$. Then $M(s,u) = u^3 G(\frac{u}{s})$.

The proof is same as theorem 3.3.

Conclusion:

In this paper, we state the definition of RAHMOH transform regarding to distributional functions and we calculate it for Dirac delta function and its derivatives. After that we proof some theorems regarding to RAHMOH,Laplace, Sumudu, and Natural transforms and the duality between them and RAHMOH transform.

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