Innovations

# On the Fractional Triple Aboodh Transform and its Properties

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***Abstract:****In this work we introduce and prove the different properties and theorems of the fractional triple Aboodh transform like the linearity property, the first and the second shifting properties, the convolution theorem, the periodic function property and the operational formula. We also give an application of this new concept to solve a factional partial differential equation in three dimensions satisfying given initial and boundary value conditions.*

***Keywords:*** *Fractional triple Aboodh transform, Inverse triple Aboodh transform, partial differential equations, Upadhyaya transform, Aboodh transform.*

## 1.Introduction

In the past two centuries, the integral transforms have been widely applied as a tool to solve various problems in pure and applied mathematics. Several integral transforms such as the most famous one introduced by P.S. Laplace (1749-1827) in 1782, called the Laplace transforms [**5**,**2**] is defined by,

$$\mathscr{L}[f(t)] = F(u) = \int_0^\infty e^{-ut} f(t)dt \qquad (1.1)$$

In the early 2011, Tarig M. Elzaki [**7**] introduced the modified Laplace transform, called the Elzaki transform (see also [**8**,**6**]), which is defined for a function of exponential order. Consider a function in the set $S$ defined as

$$S = \{f(t): \exists M, k_1, k_2 > 0, |f(t)| < Me^{\frac{|t|}{k_j}}, \text{if} t \in (-1)^j \times [0,\infty), j = 1,2\}$$

For a given function $f(t)$ in the set $S$ the constant $M$ must be finite, the numbers $k_1, k_2$maybe finite or infinite. The modified Laplace transform, i.e, the Aboodh transform denoted by the operators $A$ is defined by

$$A[f(\tau)] = K(\rho) = \frac{1}{\rho}\int_0^\infty e^{-\rho\tau} f(\tau)d\tau \qquad (1.2)$$

The variable $\rho$ in this transform is used to factorize the variable $\tau$.

The triple Aboodh transform of a function $f(x,y,\tau)$ of three variables $x$, $y$and $\tau$ , that can be expressed as a convergent infinite series, and for $(x,y,\tau) \in \mathbb{R}_3^+$ defined in the first octant of the $xy\tau$ -plane is defined by the triple modified laplace transform in the form [**16**]

$$A_{xyt}f(x,y,\tau) = K(\sigma,\rho,\delta) = \frac{1}{\sigma\rho\delta}\int_0^\infty\int_0^\infty\int_0^\infty e^{-(\sigma x+\rho y+\delta\tau)} f(x,y,t)dxdyd\tau \quad (1.3)$$

We mention here that theAboodh transform defined by Eq. (1.2) follows as the special case of the very recently introduced and the most powerful and versatile generalization of the Laplace transform, called the Upadhyaya transform (see, Upadhyaya [**19, (2.2), (2.3), p.473**]). We point out below the connection between the Upadhyaya transform and the Aboodh transform in terms of the notation of Upadhyaya [**19**, subsection **4.5**, pp.476-477] as

$$\mathcal{U}\left\{f(\tau), v, \frac{1}{v}\right\} = u\left(v, \frac{1}{v}, 1\right) = A[f(\tau), v] = K(\rho) \quad (1.4)$$

It is also to be noted here that the triple Aboodh transform Eq. (1.3) introduced early this year by triple Aboodh [**16**], is also a particular case of the Triple Upadhyaya Transform (TUT) (see, Upadhyaya [**19**, subsection **6.14**, p.501]) and the relation between the two is given by:

$$\mathcal{U}_3\left\{f(x,y,\tau); \sigma, \frac{1}{\sigma}, 1, \rho, \frac{1}{\rho}, 1, \delta, \frac{1}{\delta}, 1\right\} = u_3\left(\sigma, \frac{1}{\sigma}, 1, \rho, \frac{1}{\rho}, 1, \delta, \frac{1}{\delta}, 1\right) \quad (1.5)$$
$$A[f(x,y,\tau); \sigma,\rho,\delta] = K(\sigma,\rho,\delta)$$

As the above work of Upadhyaya [**19**] opens up many new future directions of work and applications of the Upadhyaya transform, we propose to take up the further study and applications of the Upadhyaya transform in our future works. For our present considerations the structure of this paper is organized as follows: first we begin with some basic definition of Fractional Calculus in section **2**, then define the fractional triple Aboodh transform in the Definition **3.1** in section 3 and the prove the linearity property, the convolution theorem, the first and second shitting properties, the periodic function property, and the operational formula (differential property) of this new transform in the same section. In the section **4** we obtain the exact solution of a fractional partial differential equation in three dimensions satisfying some initial value and boundary condition as an application of the results developed in section **3** and finally the conclusions are stated in section **5**.

## 2.Fundamental concepts of fractional calculus

**Definition 2.1.**[**14,8**] Let $g(x)$ be a continuous function and not necessarily differentiable function, where $\lambda > 0$ denoted a constant discretization span, the fractional difference of $g(x)$ is known as

$$\Delta^\alpha g(x) = \sum_{k=0}^{\infty}(-1)^k\binom{\alpha}{k} g[x+k\lambda] \text{for } 0 < \alpha > 1 \qquad (2.1)$$

where$\binom{\alpha}{k} = \frac{\alpha!}{k!(\alpha-k)!}$ and the $\alpha$-derivative of $g(x)$ is known as

$$g^{(\alpha)}(x) = \lim_{\lambda \to 0} \frac{\Delta_\lambda^\alpha g(x)}{\lambda^\alpha}$$

See the details in [**14,8**].

**Definition 2.2.[17]** let $g(x)$ be a continuous function, but not necessarily differentiable, then

(i). Let us assume that $g(x) = \lambda$ where $\lambda$ is constant, thus$\alpha$ -derivative of the function $g(x)$ is

$$D_x^\alpha \lambda = \begin{cases} \lambda \dfrac{x^\alpha}{\Gamma(1+\alpha)}, & \alpha > 0, \\ 0, & \text{otherwise.} \end{cases}$$

On the other hand, when $g(x) \neq \lambda$ then

$$g(x) = g(0) + \big(g(x) - g(0)\big),$$

and the fractional derivative of the function $g(x)$ is given as

$$D^\alpha g(x) = D_x^\alpha g(0) + D_x^\alpha\big(g(x) - g(0)\big),$$

(ii). For any $(\alpha > 0)$ one has

$$D^{-\alpha} g(x) = J^\alpha g(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x - \mathcal{J})^{\alpha-1} g(\mathcal{J}) d\mathcal{J} \ , \alpha > 0. \qquad (2.2)$$

**Definition 2.3.**[12] the Caputo fractional derivative of the left sided $g \in C_{-1}^n, n \in \mathbb{N} \cup \{0\}$ is defined by

$$D^\alpha g(\mathcal{J}) = \frac{\partial^\alpha g(\mathcal{J})}{\partial \mathcal{J}^\alpha} = J^{m-\alpha}\left[\frac{\partial^m g(\mathcal{J})}{\partial \mathcal{J}^m}\right], \ m - 1 < \alpha \leq m, m < N. (2.3)$$

We record properties of the operator $J^\alpha$ (see [11])

(i). $J^\alpha J^\beta g(\mathcal{J}) = J^{\alpha+\beta} g(\mathcal{J}), \quad \alpha, \beta \geq 0$

(ii). $J^\alpha \mathcal{J}^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\alpha+\mu+1)} \mathcal{J}^{\alpha+\mu} \ , \ \alpha > 0, \ \mu > -1, \mathcal{J} > 0$

(iii). $J^\alpha\big(D_*^\alpha g(\mathcal{J})\big) = g(\mathcal{J}) - \sum_{k=0}^{n-1} g^k(0^+) \frac{\mathcal{J}^\kappa}{k!}$,

**Definition 2.4.**Let be a continuous function, so the solution of the fractional differential equation

$$dy = g(x)(dx)^\alpha, \quad x \geq 0, \qquad y(0) = 0 \ , \alpha > 0,$$

By integration with respect to $(dx)^\alpha$ is the folloing

$$y(x) = \int_0^x g(\mathcal{J})(d\mathcal{J})^\alpha \ , \ y(0) = 0,$$

i.e.,

$$y(x) = \alpha \int_0^x (x - \mathcal{J})^{\alpha-1} g(\mathcal{J}) d\mathcal{J} \qquad 0 < \alpha < 1 \qquad (2.4)$$

For example, if $g(x) = x^\beta$ one obtains:

$$\int_0^x \mathcal{J}^\beta (d\mathcal{J})^\alpha = \frac{\Gamma(\beta+1)\Gamma(1+\alpha)}{\Gamma(\beta+\alpha+1)} x^{\beta+\alpha}, \ \ 0 < \alpha < 1.$$

**Definition 2.5.**[12] if then the fractional double Aboodh transform of the fractional derivative is,

$A_{x\tau}[D_*^{\alpha} g(x,)] = \rho^{\alpha} K_{\alpha}^{2}\,(x,\rho) - \sum_{k=0}^{m-1} \frac{1}{\rho^{2-\alpha+k}} g^{(k)}(x,0), \quad m-1 < \alpha \leq m,$(2.5)

**3. Theorems and properties of the fractional triple Aboodh transform:**

We define the fractional triple Aboodh transform of the functions dependent on three variables and give some properties for the same as pointed out earlier in the abstract of the paper andalso, in the section **1**above.

**Definition 3.1.** The fractional triple Aboodh transform of the function $f(x,y,\tau)$ of three variables $x,y,\tau$is defined as follows:

$$A_{xy\tau} f(x,y,\tau) = K_{\alpha}^{3}(\sigma,\rho,\delta)$$
$$= \frac{1}{\sigma^{\alpha}\rho^{\alpha}\delta^{\alpha}} \int_0^{\infty}\int_0^{\infty}\int_0^{\infty} A_{\alpha}[-(\sigma x+\rho y+\delta\tau)^{\alpha}] f(x,y\,,\tau)(dx)^{\alpha}(dy)^{\alpha}(d\tau)^{\alpha}$$

$$= \left(\frac{1}{\sigma^{\alpha}\rho^{\alpha}\delta^{\alpha}}\right) \lim_{\substack{N\to\infty\\ M\to\infty\\ K\to\infty}} \int_0^{K}\int_0^{M}\int_0^{N} A_{\alpha}[-(\sigma x+\rho y+\delta\tau)^{\alpha}] f(x,y\,,\tau)(dx)^{\alpha}(dy)^{\alpha}(d\tau)^{\alpha} \quad (3.1)$$

where$\sigma,\rho,\delta \in C$ $,x,y,\tau > 0, and A_{\alpha}(x) = \sum_{m=0}^{\infty} \frac{\Gamma(\alpha m+1)}{x^m}$ is the Mittag-Leffler function.

**Definition 3.2.** [17] Let $f(x,y,\tau)$denote a function which vanishes for negative values of $x,y,\tau$.Itstriple Laplace's transform of order $\alpha$(or its $\alpha^{th}$ fractional Laplace transform) is defined by the followingexpression:

$$L_{xy\tau} f(x,y,\tau) = F_{\alpha}^{3}(\sigma,\rho,\delta) = \int_0^{\infty}\int_0^{\infty}\int_0^{\infty} A_{\alpha}[-(\sigma x+\rho y+\delta\tau)^{\alpha}] f(x,y\,,\tau)(dx)^{\alpha}(dy)^{\alpha}(d\tau)^{\alpha}$$
(3.2)

$$= \lim_{\substack{N\to\infty\\ M\to\infty\\ K\to\infty}} \int_0^{K}\int_0^{M}\int_0^{N} A_{\alpha}[-(\sigma x+\rho y+\delta\tau)^{\alpha}] f(x,y\,,\tau)(dx)^{\alpha}(dy)^{\alpha}(d\tau)^{\alpha}$$

**Theorem 3.3.The Linearity of the triple fractional Aboodh transform:**

Let $f(x,y,\tau)$ and$g(x,y,\tau)$be functions whose triple fractional Aboodhtransforms exist, then

$$A_{xy\tau}[\theta f(x,y,\tau) + \beta g(x,y,\tau)] = \theta A_{xy\tau}[f(x,y,\tau)] + \beta A_{xy\tau}[g(x,y,\tau)]$$

where$\theta$ and $\beta$are constants.

**Proof.**

$A_{xy\tau}[\theta f(x,y,\tau)+\beta g(x,y,\tau)]$

$$=\frac{1}{\sigma^{\alpha}\rho^{\alpha}\delta^{\alpha}}\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}[\theta f(x,y,\tau)+\beta g(x,y,\tau)]A_{\alpha}[-(\sigma x+\rho y+\delta\tau)^{\alpha}](dx)^{\alpha}(dy)^{\alpha}(d\tau)^{\alpha}$$

$$=\frac{1}{\sigma^{\alpha}\rho^{\alpha}\delta^{\alpha}}\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}[\theta f(x,y,\tau)]A_{\alpha}[-(\sigma x+\rho y+\delta\tau)^{\alpha}](dx)^{\alpha}(dy)^{\alpha}(d\tau)^{\alpha}$$

$$+\frac{1}{\sigma^{\alpha}\rho^{\alpha}\delta^{\alpha}}\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}[\beta g(x,y,\tau)]A_{\alpha}[-(\sigma x+\rho y+\delta\tau)^{\alpha}](dx)^{\alpha}(dy)^{\alpha}(d\tau)^{\alpha}$$

$$=\frac{1}{\sigma^{\alpha}\rho^{\alpha}\delta^{\alpha}}\theta\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}[f(x,y,\tau)]A_{\alpha}[-(\sigma x+\rho y+\delta\tau)^{\alpha}](dx)^{\alpha}(dy)^{\alpha}(d\tau)^{\alpha}+$$

$$\frac{1}{\sigma^{\alpha}\rho^{\alpha}\delta^{\alpha}}\beta\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}[g(x,y,\tau)]A_{\alpha}[-(\sigma x+\rho y+\delta\tau)^{\alpha}](dx)^{\alpha}(dy)^{\alpha}(d\tau)^{\alpha}$$

$$=\theta A_{xy\tau}[f(x,y,\tau)]+\beta A_{xy\tau}[g(x,y,\tau)]$$

**Theorem 3.4.The First Shifting Property:**

If $A_{xy\tau}[f(x,y,\tau)]=K_{\alpha}^{3}(\sigma,\rho,\delta)$,then

$$A_{xy\tau}\big(A_{\alpha}[-(\sigma\theta x+\rho\beta y+\delta k\tau)^{\alpha}]f(x,y,\tau)\big)=K_{\alpha}^{3}(1+\theta,1+\beta,1+k)$$

**Proof:**

Let

$$A_{xy\tau}[f(x,y,\tau)]=K_{\alpha}^{3}(\sigma,\rho,\delta)$$

$$=\frac{1}{\sigma^{\alpha}\rho^{\alpha}\delta^{\alpha}}\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}A_{\alpha}[-(\sigma x+\rho y+\delta\tau)^{\alpha}]\,f(x,y,\tau)(dx)^{\alpha}(dy)^{\alpha}(d\tau)^{\alpha}$$

Then

$$A_{xy\tau}\big(A_{\alpha}[-(\sigma\theta x+\rho\beta y+\delta k\tau)^{\alpha}]f(x,y,\tau)\big)$$

$$=\frac{1}{\sigma^{\alpha}\rho^{\alpha}\delta^{\alpha}}\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}A_{\alpha}\,[-(\sigma x+\rho y+\delta\tau)^{\alpha}]\big[A_{\alpha}[-(\sigma\theta x+\rho\beta y+\delta k\tau)^{\alpha}]\big]f(x,y,\tau)(dx)^{\alpha}(dy)^{\alpha}(d\tau)^{\alpha}$$

by using the equality $A_{\alpha}[\mu(\sigma x+\rho y+\delta\tau)^{\alpha}]=A_{\alpha}\mu(\sigma x)^{\alpha}A_{\alpha}\mu(\rho y)^{\alpha}A_{\alpha}\mu(\delta\tau)^{\alpha}$

which implies that

$$A_{xy\tau}[A_{\alpha}[-(\sigma\theta x+\rho\beta y+\delta k\tau)^{\alpha}]f(x,y,\tau)]$$

$$= \frac{1}{\sigma^\alpha \rho^\alpha \delta^\alpha} \int_0^\infty \int_0^\infty \int_0^\infty A_\alpha[-(\sigma(1+\theta)x + \rho(1+\beta)y + \delta(1 + k\,)\tau)^\alpha] f(x,y,\tau)(dx)^\alpha (dy)^\alpha (d\tau)^\alpha$$

$$= \frac{1}{\sigma^\alpha} \int_0^\infty A_\alpha\,[-(\sigma(1 + \theta)x)^\alpha] \left\{ \frac{1}{\rho^\alpha \delta^\alpha} \int_0^\infty A_\alpha[-(\rho(1+\beta)y + \delta(1 + k\,)\tau)^\alpha] f(x,y,\tau)(dy)^\alpha (d\tau)^\alpha \right\} (dx)^\alpha$$

$$= \frac{1}{\sigma^\alpha} \int_0^\infty A_\alpha\,[-(\sigma(1+\theta)x)^\alpha] f(x, 1+\beta, 1+k)dx$$

$$= K_\alpha^3(1+\theta, 1+\beta, 1+k).$$

**Theorem 3.5.The Periodic Property:**

If $f(x,y,\tau)$is a periodic function of periods $\theta$, $\beta$and krespectively, in the variables $x, y$ and $\tau$ , i.e.,

$f(x\ +\theta\ , y\ +\beta\ , \tau\ +\ k)\ =\ f(x,y,\tau)$and if$A_{xy\tau}[f(x,y,\tau)]$

exists then

$$A_{xy\tau}[f(x,y,\tau)] = K_\alpha^3(\sigma,\rho,\delta)$$

$$= \frac{1}{\sigma^\alpha \rho^\alpha \delta^\alpha \big(1 - [A_\alpha[-(\sigma\theta + \rho\beta + \delta k)^\alpha]]\big)} \int_0^\theta \int_0^\beta \int_0^k A_\alpha[-(\sigma x + \rho y + \delta\tau)^\alpha]\, f(x,y,\tau)(dx)^\alpha (dy)^\alpha (d\tau)^\alpha.$$

**Proof:**

Let

$$A_{xy\tau}[f(x,y,\tau)] = \ K_\alpha^3(\sigma,\rho,\delta)$$

$$= \frac{1}{\sigma^\alpha \rho^\alpha \delta^\alpha} \int_0^\infty \int_0^\infty \int_0^\infty A_\alpha[-(\sigma x + \rho y + \delta\tau)^\alpha]\, f(x,y,\tau)(dx)^\alpha (dy)^\alpha (d\tau)^\alpha$$

$$= \frac{1}{\sigma^\alpha \rho^\alpha \delta^\alpha} \int_0^\theta \int_0^\beta \int_0^k A_\alpha[-(\sigma x + \rho y + \delta\tau)^\alpha]\, f(x,y,\tau)(dx)^\alpha (dy)^\alpha (d\tau)^\alpha + \frac{1}{\sigma^\alpha \rho^\alpha \delta^\alpha} \int_\alpha^\infty \int_\beta^\infty \int_k^\infty A_\alpha[-(\sigma x + \rho y + \delta\tau)^\alpha]\, f(x,y,\tau)(dx)^\alpha (dy)^\alpha (d\tau)^\alpha$$

Putting $x\ =\ u+\theta, y\ =\ v+\beta, \tau\ =\ w+k$ in the second triple integral we get

$$A_{xy\tau}[f(x,y,\tau)] = K_\alpha^3(\sigma,\rho,\delta)$$

$$= \frac{1}{\sigma^\alpha \rho^\alpha \delta^\alpha} \int_0^\theta \int_0^\beta \int_0^k A_\alpha[-(\sigma x + \rho y + \delta\tau)^\alpha]\, f(x,y,\tau)(dx)^\alpha (dy)^\alpha (d\tau)^\alpha + \frac{1}{\sigma^\alpha \rho^\alpha \delta^\alpha} \int_\alpha^\infty \int_\beta^\infty \int_k^\infty A_\alpha[-(\sigma(u+\theta) + \rho(v+\beta) + \delta(w+k))^\alpha]\, f(u+\theta, v+\beta, w + k)(du)^\alpha (dv)^\alpha (dw)^\alpha$$

Or,

$$K_{\alpha}^{3}(\sigma,\rho,\delta)=\frac{1}{\sigma^{\alpha}\rho^{\alpha}\delta^{\alpha}}\int_{0}^{\theta}\int_{0}^{\beta}\int_{0}^{k}A_{\alpha}\left[-(\sigma x+\rho y+\delta\tau)^{\alpha}\right]f(x,y,\tau)(dx)^{\alpha}(dy)^{\alpha}(d\tau)^{\alpha}+$$

$$\frac{1}{\sigma^{\alpha}\rho^{\alpha}\delta^{\alpha}}(A_{\alpha}[-(\sigma\theta+\rho\beta+\delta k))^{\alpha}])\int_{\alpha}^{\infty}\int_{\beta}^{\infty}\int_{k}^{\infty}A_{\alpha}[-(\sigma\theta+\rho\beta)+\delta k)^{\alpha}]f(u+\theta,v+\beta,w+k)(du)^{\alpha}(dv)^{\alpha}(dw)^{\alpha}$$

$$=\frac{1}{\sigma^{\alpha}\rho^{\alpha}\delta^{\alpha}}\int_{0}^{\theta}\int_{0}^{\beta}\int_{0}^{k}A_{\alpha}[-(\sigma x+\rho y+\delta\tau)^{\alpha}]f(x,y,\tau)(dx)^{\alpha}(dy)^{\alpha}(d\tau)^{\alpha}+$$

$$\frac{1}{\sigma^{\alpha}\rho^{\alpha}\delta^{\alpha}}\left[A_{\alpha}[-(\sigma\theta+\rho\beta+\delta k))^{\alpha}]\right]\int_{0}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty}A_{\alpha}[-(\sigma u+\rho v+\delta w)^{\alpha}]f(u,v,w)(du)^{\alpha}(dv)^{\alpha}(dw)^{\alpha}$$

$$=\frac{1}{\sigma^{\alpha}\rho^{\alpha}\delta^{\alpha}}\int_{0}^{\theta}\int_{0}^{\beta}\int_{0}^{k}A_{\alpha}[-(\sigma x+\rho y+\delta\tau)^{\alpha}]f(x,y,\tau)(dx)^{\alpha}(dy)^{\alpha}(d\tau)^{\alpha}+\left[A_{\alpha}[-(\sigma\theta+\rho\beta)+\delta k)^{\alpha}]\right]K_{\alpha}^{3}(\sigma,\rho,\delta)$$

Therefore,

$$\frac{1}{\sigma^{\alpha}\rho^{\alpha}\delta^{\alpha}}\int_{0}^{\theta}\int_{0}^{\beta}\int_{0}^{k}A_{\alpha}[-(\sigma x+\rho y+\delta\tau)^{\alpha}]f(x,y,\tau)(dx)^{\alpha}(dy)^{\alpha}(d\tau)^{\alpha}=K_{\alpha}^{3}(\sigma,\rho,\delta)-\left[A_{\alpha}[-(\sigma\theta+\rho\beta+\delta k)^{\alpha}]\right]K_{\alpha}^{3}(\sigma,\rho,\delta).$$

Hence,

$$K_{\alpha}^{3}(\sigma,\rho,\delta)=\frac{1}{\sigma^{\alpha}\rho^{\alpha}\delta^{\alpha}\left(1-\left[A_{\alpha}[-(\sigma\theta+\rho\beta+\delta k)^{\alpha}]\right]\right)}\int_{0}^{\alpha}\int_{0}^{\beta}\int_{0}^{k}A_{\alpha}[-(\sigma x+\rho y+\delta\tau)^{\alpha}]f(x,y,\tau)(dx)^{\alpha}(dy)^{\alpha}(d\tau)^{\alpha}$$

**Theorem 3.6.The Second Shifting Property:**

If$A_{xy\tau}[f(x,y,\tau)]=K_{\alpha}^{3}(\sigma,\rho,\delta)$

then,

$$A_{xy\tau}[f(x-\theta,y-\beta,\tau-k)H(x-\theta,y-\beta,\tau-k)]=A_{\alpha}[-(\sigma\alpha+\rho\beta+\delta k)^{\alpha}]K_{\alpha}^{3}(\sigma,\rho,\delta)$$

where$H(x,y,\tau)$ is the Heaviside unit step function defined by

$$H(x-\theta,y-\beta,\tau-k)=\begin{cases}1, & when, x>\theta, y>\beta, \tau>k\\ 0, & when, x<\theta, y<\beta, \tau<k.\end{cases}$$

**Proof:**

Let

$$A_{xy\tau}[f(x,y,\tau)]=K_{\alpha}^{3}(\sigma,\rho,\delta)$$

$$= \frac{1}{\sigma^\alpha \rho^\alpha \delta^\alpha} \int_0^\infty \int_0^\infty \int_0^\infty A_\alpha[-(\sigma x + \rho y + \delta\tau)^\alpha] f(x,y,\tau)(dx)^\alpha (dy)^\alpha (d\tau)^\alpha.$$

Then

$$A_{xy\tau}[f(x-\theta, y-\beta, \tau-k)H(x-\theta, y-\beta, \tau-k)] =$$

$$= \frac{1}{\sigma^\alpha \rho^\alpha \delta^\alpha} \int_0^\infty \int_0^\infty \int_0^\infty A_\alpha[-(\sigma x + \rho y + \delta\tau)^\alpha] f(x-\theta, y-\beta, \tau-k)H(x-\theta, y-\beta, \tau - k)(dx)^\alpha (dy)^\alpha (d\tau)^\alpha$$

$$= \frac{1}{\sigma^\alpha \rho^\alpha \delta^\alpha} \int_\alpha^\infty \int_\beta^\infty \int_k^\infty A_\alpha[-(\sigma x + \rho y + \delta\tau)^\alpha] f(x-\theta, y-\beta, \tau-k)(dx)^\alpha (dy)^\alpha (d\tau)^\alpha$$

which, on putting $x-\theta = u, y-\beta = v, \tau-k = w$ gives

$$A_{xy\tau}[f(x-\theta, y-\beta, \tau-k)H(x-\theta, y-\beta, \tau-k)]$$

$$= \left[A_\alpha[-(\sigma\theta + \rho\beta + \delta k)^\alpha]\right] \frac{1}{\sigma^\alpha \rho^\alpha \delta^\alpha} \int_0^\infty \int_0^\infty \int_0^\infty A_\alpha[-(\sigma u + \rho v + \delta w)^\alpha] f(u,v,w)(du)^\alpha (dv)^\alpha (dw)^\alpha$$

$$= \left[A_\alpha[-(\sigma\theta + \rho\beta + \delta k)^\alpha]\right] K_\alpha^3(\sigma, \rho, \delta).$$

**Theorem 3.7.The Convolution Theorem:**

If

$$A_{xy\tau}[F(x,y,\tau)] = f_\alpha^3(\sigma,\rho,\delta), A_{xy\tau}[G(x,y,\tau)] = g_\alpha^3(\sigma,\rho,\delta)$$

then the convolution of the functions $F(x,y,\tau)$ $and G(x,y,\tau)$ is denoted by $F *** G$and is defined by

$$A_{xy\tau}[(F *** G)(x,y,\tau)] = \frac{1}{\sigma^\alpha \rho^\alpha \delta^\alpha} \int_0^x \int_0^y \int_0^\tau F(x-\theta, y-\beta, \tau-k)G(\theta,\beta,k)(dx)^\alpha (dy)^\alpha (d\tau)^\alpha$$

and we have

$$A_{xy\tau}[(F *** G)(x,y,\tau)] = A_{xy\tau}[F(x,y,\tau)] \cdot A_{xy\tau}[G(x,y,\tau)] = f_\alpha^3(\sigma,\rho,\delta) \cdot g_\alpha^3(\sigma,\rho,\delta)$$

**Proof:**

From the definition of the convolution, we have

$$A_{xy\tau}[(F *** G)(x,y,\tau)] = \frac{1}{\sigma^\alpha\rho^\alpha\delta^\alpha}\int_0^\infty\int_0^\infty\int_0^\infty A_\alpha[-(\sigma x+\rho y+\delta\tau)^\alpha]\,(F ** * G)(x,y,\tau)(dx)^\alpha(dy)^\alpha(d\tau)^\alpha$$

$$= \frac{1}{\sigma^\alpha\rho^\alpha\delta^\alpha}\int_0^\infty\int_0^\infty\int_0^\infty A_\alpha[-(\sigma x+\rho y+\delta\tau)^\alpha]\times \left[\frac{1}{\sigma^\alpha\rho^\alpha\delta^\alpha}\int_0^x\int_0^y\int_0^\tau \quad F(x-\theta,y-\beta,\tau-k)G(\alpha,\beta,k)(d\theta)^\alpha(d\beta)^\alpha(dk)^\alpha\right](dx)^\alpha(dy)^\alpha(d\tau)^\alpha$$

which on using the Heaviside unit step function yields

$$A_{xy\tau}[(F *** G)(x,y,\tau)] = \frac{1}{\sigma^\alpha\rho^\alpha\delta^\alpha}\int_0^\infty\int_0^\infty\int_0^\infty G(\theta,\beta,k)\,(d\theta)^\alpha(d\beta)^\alpha(dk)^\alpha\times \left[\frac{1}{\sigma^\alpha\rho^\alpha\delta^\alpha}\int_0^x\int_0^y\int_0^\tau A_\alpha[-(\sigma x+\rho y+\delta\tau)^\alpha]\,F(x-\theta,y-\beta,\tau-k)H(x-\theta,y-\beta,\tau-k\,)\right](dx)^\alpha(dy)^\alpha(d\tau)^\alpha$$

The above expression may be simplified by using the result of the Theorem **3.6**

$$A_{xy\tau}[(F *** G)(x,y,\tau)]$$
$$= \frac{1}{\sigma^\alpha\rho^\alpha\delta^\alpha}\int_0^\infty\int_0^\infty\int_0^\infty A_\alpha[-(\sigma x+\rho y+\delta\tau)^\alpha]\,f_\alpha^3(\theta,\rho,\delta)\;G(\theta,\beta,k)(d\theta)^\alpha(d\beta)^\alpha(dk)^\alpha$$
$$= \frac{1}{\sigma^\alpha\rho^\alpha\delta^\alpha}f_\alpha^3(\theta,\rho,\delta)\int_0^\infty\int_0^\infty\int_0^\infty A_\alpha[-(\sigma x+\rho y+\delta\tau)^\alpha]\,G(\theta,\beta,k)(d\theta)^\alpha(d\beta)^\alpha(dk)^\alpha$$
$$= f_\alpha^3(\theta,\rho,\delta)\cdot g_\alpha^3(\theta,\rho,\delta).$$

**Theorem 3.8.The Operational Formula:**

Let $f(x,y,\tau)\in C^\lambda(\mathbb{R}^+\times\mathbb{R}^+\times\mathbb{R}^+)$, then the operational formula for the triple fractional Aboodh transform is given by

$$A[D_x^\alpha f(x,y,\tau):(\sigma,\rho,\delta)] = \sigma^\alpha K_\alpha^3(\sigma,\rho,\delta) \quad -\frac{1}{\sigma^\alpha}\Gamma(\alpha+1)K_\alpha^3(0,\rho,\delta) \qquad (3.3)$$

**Proof:**

$$A_{xy\tau}[f(x,y,\tau)] = K_\alpha^3(\sigma,\rho,\delta) = \frac{1}{\sigma^\alpha\rho^\alpha\delta^\alpha}\int_0^\infty\int_0^\infty\int_0^\infty A_\alpha\,[-(\sigma x+\rho y+\delta\tau)^\alpha]\,f(x,y,\tau)(dx)^\alpha(dy)^\alpha(d\tau)^\alpha$$

Then

$$A[D_x^\alpha f(x,y,\tau):(\sigma,\rho,\delta)]$$

$$= \frac{1}{\sigma^\alpha \rho^\alpha \delta^\alpha} \int_0^\infty \int_0^\infty \int_0^\infty A_\alpha[-(\sigma x + \rho y + \delta\tau)^\alpha]\, f^{(\alpha)}(x,y,\tau)(dx)^\alpha (dy)^\alpha (d\tau)^\alpha$$

$$= \frac{1}{\rho^\alpha \delta^\alpha} \int_0^\infty A_\alpha[-(\delta\tau)^\alpha]\left[\int_0^\infty A_\alpha[-(\rho y)^\alpha]\left[\frac{1}{\sigma^\alpha}\int_0^\infty A_\alpha[-(\sigma x)^\alpha] f^{(\alpha)}(x,y,\tau)(dx)^\alpha\right]\right](dy)^\alpha (d\tau)^\alpha.$$

Applying the integration by parts to the expressions inside the square brackets on the right-hand side of the above equation we have

$$A[D_x^\alpha f(x,y,\tau):(\sigma,\rho,\tau)]$$

$$= \frac{1}{\rho^\alpha \delta^\alpha} \int_0^\infty A_\alpha[-(\delta\tau)^\alpha]\left[\int_0^\infty A_\alpha\,[-(\rho y)^\alpha]\left\{\frac{1}{\sigma^\alpha}\left[\Gamma(\alpha+1) f(x,y,\tau) A_\alpha[-(\sigma x)^\alpha]\right]_0^\infty + \frac{1}{\sigma^\alpha}\int_0^\infty A_\alpha[-(\sigma x)^\alpha] f(x,y,\tau)(dx)^\alpha\right\}\right](dy)^\alpha (d\tau)^\alpha$$

$$= \frac{1}{\rho^\alpha \delta^\alpha} \int_0^\infty A_\alpha[-(\delta\tau)^\alpha]\left[\int_0^\infty A_\alpha\,[-(\rho y)^\alpha]\left\{\frac{-1}{\sigma^\alpha}\left[\Gamma(\alpha+1) f(0,y,\tau) + \frac{1}{\sigma^\alpha}\int_0^\infty A_\alpha[-(\sigma x)^\alpha] f(x,y,\tau)(dx)^\alpha\right]\right\}\right](dy)^\alpha (d\tau)^\alpha$$

$$= \frac{1}{\rho^\alpha \delta^\alpha} \int_0^\infty A_\alpha[-(\delta\tau)^\alpha]\left[\int_0^\infty A_\alpha\,[-(\rho y)^\alpha]\left[\int_0^\infty A_\alpha[-(\sigma x)^\alpha]\, f(x,y,\tau)(dx)^\alpha - \frac{1}{\sigma^\alpha}\Gamma(\alpha+1) f(0,y,\tau)\right]\right](dy)^\alpha (d\tau)^\alpha$$

$$= \sigma^\alpha K_\alpha^3(\sigma,\rho,\delta) \quad - \frac{1}{\sigma^\alpha}\Gamma(\alpha+1) K_\alpha^3(0,\rho,\delta)$$

## 4. Applications

In this section on the assumption that the inverse fractional triple Aboodh transform exists, we use the inverse fractional triple Aboodh transform to obtain the exact solutions of the partial differential equations of fractional order in three dimensions with initial and boundary conditions.

**Example 4.1.**

Consider the following partial differential equation of fractional order

$$D_\tau^\alpha f(x,y,\tau) = \frac{\partial^2 f(x,y,\tau)}{\partial x^2}, \quad 0 < \alpha \le 1 \qquad (4.1)$$

with the following initial and boundary conditions

$$f(0,y,\tau) = 0, f_x(0,y,\tau) = cosyA_\alpha(-\tau^\alpha)$$
$$f(x,y,0) = cosxcosy$$

**Solution.**

Taking the fractional triple Aboodh transform of Eq. (**4.1**) and the fractional double Aboodh transform of the initial and the boundary conditions gives

$$A_{xy\tau}[D_\tau^\alpha f(x,y,\tau)] = A_{xy\tau}\left[\frac{\partial^2 f(x,y,\tau)}{\partial x^2}\right]$$

$$\begin{aligned}&\delta^\alpha A_{xy\tau}[f(x,y,\tau)] - \frac{1}{\delta^\alpha}\Gamma(\alpha+1)A_{xy\tau}[f(x,y,0)]\\ &= \sigma^2 A_{xy\tau}[f(x,y,\tau)] - A_{xy\tau}[f(0,y,\tau)] - \frac{1}{\sigma}A_{xy\tau}\left[\frac{\partial f(0,y,\tau)}{\partial x}\right]\end{aligned} \quad (4.2)$$

$$K_\alpha^3(\sigma,\rho,0) = \frac{1}{\sigma(1+\sigma^2)}\frac{1}{(1+\rho^2)}, K_\alpha^3(0,\rho,\delta) = 0, \frac{\partial K_\alpha^3(0,\rho,\delta)}{\partial x} = \frac{1}{(1+\rho^2)}\frac{\Gamma(\alpha+1)}{\delta^\alpha(1+\delta^2)} \quad (4.3)$$

Then Eq. (**4.2**) becomes

$$A_{xy\tau}[f(x,y,\tau)](\delta^\alpha - \sigma^2) = \frac{1}{\delta^\alpha}\Gamma(\alpha+1)\frac{1}{\sigma(1+\sigma^2)}\frac{1}{(1+\rho^2)} - \frac{1}{\sigma}\frac{1}{(1+\rho^2)}\frac{\Gamma(\alpha+1)}{\delta^\alpha(1+\delta^\alpha)}$$

$$A_{xy\tau}[f(x,y,\tau)](\delta^\alpha - \sigma^2) = \frac{(\delta^\alpha - \sigma^2)\Gamma(\alpha+1)}{\sigma\delta^\alpha(1+\sigma^2)(1+\rho^2)(1+\delta^\alpha)}$$

$$A_{xy\tau}[f(x,y,\tau)] = \frac{\Gamma(\alpha+1)}{\sigma\delta^\alpha(1+\sigma^2)(1+\rho^2)(1+\delta^\alpha)}$$

Applying inverse fractional triple Aboodh transform, we get

$$f(x,y,\tau) = sinxcosyA_\alpha[-\tau^\alpha]$$

which is the required exact solution of Eq. (**4.1**).

## 5. Conclusion

This work introduces the definition of the fractional triple Aboodh transform and the various properties like the linearity property, the first and the second shifting properties, the periodic property, theconvolution theorem and the operational formula are deduced and the results obtained are applied to find the exact solution of a fractional partial differential equation in three dimensions satisfying some initial and boundary value conditions.

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